HYDRAULICS

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HYDRAULICS

CHAPTER I

STREAM LINES

Stream Lines of a Perfect Fluid

§ 1. **Stream Line Defined.**—A stream line is the line, whether straight or curved, that is traced by a particle in a current of liquid. In steady motion, each individual stream line preserves its figure and position unchanged, and marks the path of a track or filament, or a continuous series of particles that follow each other. The directions of the motions in different parts of a steady current may be represented to the eye by drawing the group of stream lines traced by different particles in this current. Thus, a current may be conceived to be divided by insensibly thin partitions, following the course of the stream lines, into a number of elementary streams; and the positions of these partitions may be conceived to be so adjusted that the volumes of flow in all the elementary streams is the same. In that case, the velocity of a particle at any point is inversely proportional to the area of the transverse section through that point of the elementary stream to which the particle belongs; or, in plane motion, inversely proportional to the distance between the stream lines at that point. Thus, the stream lines will show, not only the direction of the motion, but the velocity.

§ 2. **Equations of Motion of a Perfect Fluid.**—The assumptions made are—

(1) That the motion is steady, and that there are no forces of acceleration or retardation;

(2) That the fluid is perfect, so that at a point the pressure is the same in all directions;

(3) That viscous resistances may be disregarded;

(4) That the fluid is incompressible.

Let a = cross-sectional area of the stream tube in square feet;

p = intensity of pressure at that point in pounds per square foot;

v = velocity in feet per second;

z = elevation above some datum;

w = weight of fluid in pounds per cubic foot.

The relation between the pressure, velocity, and elevation is obtained from the energy equation.

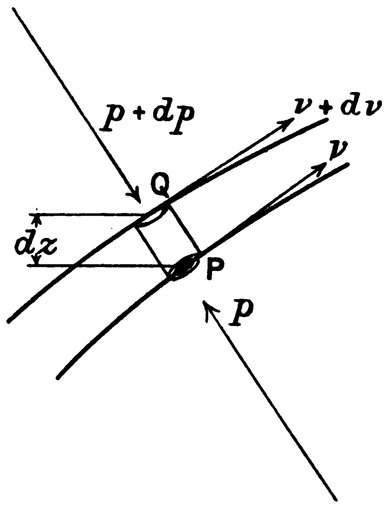

FIG. 1.

The kinetic energy per pound is $\frac{v^2}{2g}$; the pressure energy is $\frac{p}{w}$, and the potential energy is z. Thus, the total energy per pound is

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{constant} = h.$$

The equation of continuity gives

$$va = \text{constant} = Q$$

when Q is the discharge along all the stream lines, in cubic feet per second.

The stream lines are invariably curved, and, consequently, there will be a variation of pressure, in a normal direction, across two adajcent stream lines. Let Fig. 1 represent two adjacent stream tubes, suffix (1) and (2) referring to the inner and outer tubes. Along the two tubes—

$$\frac{p_2}{w} + \frac{v_2^2}{2g} + z_2 = h_2$$

$$\frac{p_1}{w} + \frac{v_1^2}{2g} + z_1 = h_1$$

where h_2 and h_1 are both constants, but not necessarily equal. Subtracting—

$$\frac{p_2 - p_1}{w} + \frac{v_2^2 - v_1^2}{2g} + z_2 - z_1 = h_2 - h_1$$

or, in the limit—

$$\frac{\delta p}{w} + \frac{v\delta v}{g} + \delta z = \delta h$$

in which δh is constant, and the increment of pressure is across the stream tubes. Since the intensity of pressure is the same in all directions, the δp in this equation is the increment of pressure radially outwards.

This radial pressure may be calculated in a second way. A small element, PQ, of unit area, has its axis along the normal to the curves. The pressure over the end sections, P and Q, are p and $p + \delta p$; the velocities at P and Q are v and $v + \delta v$; and the difference in elevation between the end section is δz—the motion being in a vertical plane. Also, let t be the thickness of the tube, namely PQ, and ρ be the radius of curvature of the stream lines at P.

The mass of water in the cylinder PQ is $\frac{wt}{g}$, its radial acceleration is $\frac{v^2}{\rho}$; therefore the intensity of pressure necessary to cause this acceleration is

$$\frac{wt}{g}\,\frac{v^2}{\rho}.$$

Hence, resolving the forces along the normal—

$$\delta p + w\delta z = \frac{wtv^2}{g\rho}.$$

Since δp has the same meaning as in the first equation, the result is

$$\delta h = \frac{vdv}{g} + \frac{tv^2}{g\rho}$$

$$= \frac{vt}{g}\left(\frac{v}{\rho} + \frac{\delta v}{t}\right).$$

Along the stream tubes, δh and vt are constant. Thus, in a perfect fluid, for equilibrium of forces—

$$\frac{v}{\rho} + \frac{\delta v}{t} = \text{constant}$$

for two adjacent lines.

§ 3. **Molecular Rotation and Distortion—Irrotational and Rotational Motion.**—The above equation has a physical interpretation. If (Fig. 2) O be the centre of curvature at the point P, $\frac{v}{\rho}$ is the angular velocity of OP about O; that is, it is the angular velocity of the tangent PT about P. Again, $\frac{\delta v}{t}$ is the angular velocity of PQ about P. The element is therefore distorted; but the resultant distortion may be resolved into two simple components. Let *ab*, *ac* be the sides of the original square (Fig. 3), and turn the square about the point *a* until it comes to the position shown by the sides *ad*, *ae*, the diagonal being *af*. Now,

imagine that, keeping af fixed, the square assumes the position of the rhombus, whose sides are ag, ah, the diagonal line being ak.

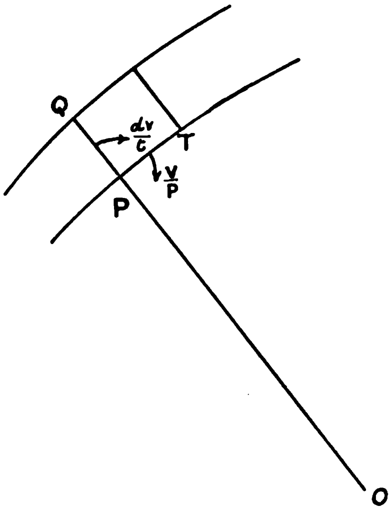

Fig. 2.

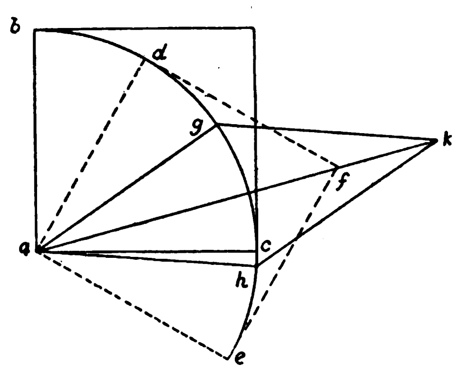

Fig. 3.

This combination is equivalent to a rotation about a point, and a pure distortion.

Now, the rate at which the angle bag increases is $\frac{dv}{t}$ radians per second, and of the angle cah, $\frac{v}{\rho}$ radians per second. Thus, the sides of the square approach at a rate of

$$\frac{dv}{t} - \frac{v}{\rho}$$

radians per second in opposite directions and at equal rates. Hence the angles dag, eah are equal to

$$\tfrac{1}{2}\left(\frac{dv}{t} - \frac{v}{\rho}\right).$$

The angles bad, cae are, therefore, equal to

$$\frac{dv}{t} - \tfrac{1}{2}\left(\frac{dv}{t} - \frac{v}{\rho}\right) = \tfrac{1}{2}\left(\frac{dv}{t} + \frac{v}{\rho}\right).$$

The motion is thus equivalent to a bodily rotation of an element about an axis of

$$\tfrac{1}{2}\left(\frac{dv}{t} + \frac{v}{\rho}\right)$$

and a pure distortion of

$$\frac{dv}{t} - \frac{v}{\rho}$$

in radians per second.

The value of

$$\tfrac{1}{2}\left(\frac{dv}{t} + \frac{v}{\rho}\right)$$

is called the *spin* or *molecular rotation*, and is constant along a stream tube (§ 2). Each of the terms $\frac{v}{\rho}$, $\frac{dv}{t}$ will vary from point to point, and so likewise will the distortion, namely, their difference; but half their sum is constant along the stream tube, but varying in different tubes.

If, for a particular stream tube, the molecular rotation is zero, the motion is said to be *irrotational*. If it is not zero, it is said to be *rotational*. In some parts, the motion might be irrotational, whilst in others it might be rotational. It is of importance to appreciate the nature of molecular rotation. Such motion does not necessarily result from revolution about a fixed axis. Spin is a rotation of a small element of the fluid about an axis drawn through a mean point inside the element.

In a perfect fluid there is no tangential force, so that the motion must have been generated from rest, or some previous motion must have been impressed upon the water.

Thus—

$$\text{the spin} = A = \tfrac{1}{2}\left(\frac{v}{\rho} + \frac{\delta v}{t}\right) = \frac{g}{2} \cdot \frac{\delta h}{vt}.$$

If the head be the same for every stream tube, $\delta h = 0$, and therefore the spin is zero, whence

$$\frac{v}{\rho} + \frac{dv}{t} = 0$$

and the motion can be generated from rest.

If the flow be in parallel streams, $\rho = \infty^{y}$ and $v =$ constant, that is, the velocity is the same at all points in the cross-section. If the motion be in circular stream lines of radius r, the equation becomes

$$\frac{v}{r} + \frac{dv}{dr} = 0$$

or $vr =$ constant.

The velocity is inversely proportional to the radius, and the motion cannot extend to the centre. It is called a *free vortex*, and if, by any slow radial motion, portions of the water strayed from one stream line to another, they would take up their positions under the existing pressure and velocity.

This represents a motion about a fixed axis, in which the *spin* is zero. The rate of angular distortion is

$$-\frac{v}{r} + \frac{dv}{dr} = -\frac{2v}{r}$$

and increases as the radius decreases.

If the distortion is zero—

$$-\frac{v}{r} + \frac{dv}{dr} = 0$$

and v and r, the water rotating as a solid body. This is called a *forced* vortex. The spin is $\frac{1}{2} \cdot \frac{2v}{r}$, and is, therefore, equal to the angular velocity about the fixed axis.

§ 4. Plotting Stream Lines in Plane Motion.—Thus, in dealing with the stream lines of a perfect fluid generated from rest, for each stream tube the following conditions must be satisfied :—

$$vt = \text{constant}$$

$$\frac{v}{\rho} + \frac{dv}{t} = 0$$

v being the velocity, ρ the radius of curvature, and t the thickness of the tube.

From the first equation—

$$tdv + vdt = 0$$

the increments referring to adjacent stream lines; and, by substitution in the second—

$$dt = \frac{t^2}{\rho}.$$

This enables the stream lines of a system to be plotted when two of the stream lines are known.

In the case of motion between fixed boundaries, or of motion round a solid, the forms of the boundaries must obviously be stream lines. Starting with one of the boundary stream lines, plot a second stream line very near to the first. If the thickness at any point and the radius of curvature be measured, the expression above gives the value of dt, and the thickness of the second stream tube will be $t + dt$. This process may be imagined to be performed until the second boundary stream is reached. If the final stream line coincide with the boundary stream line, the problem is solved. Thus, when the conditions of continuity and irrotationality are fulfilled, corresponding to given boundaries, there is one, and only one, set of stream lines; and the velocity at any point of a section bears a definite relation to the velocity at all other points of the liquid.

The same statement is true if an infinite stream of liquid flows past fixed bodies of known shapes.

§ 5. **Composition of Elementary Streams.**—If a layer of liquid is acted upon, at the same time, by two sets of forces, which, if acting separately, produce currents consisting of two different sets of streams, the combined action of these two sets of forces will produce currents consisting of a third set of elementary streams, which may be regarded as the resultants of the two former sets. The stream lines marking the boundaries of the first two sets of elementary streams may be called the *component* stream lines; and those marking the boundaries of the third set, the *resultant* stream lines. The principle which connects the resultant streams with the component streams is as follows:—

The resultant stream lines pass diagonally through all the angles of the network formed by the component stream lines.

Let (Fig. 4) AA, A′A′ be a pair of stream lines belonging to

one set, and BB, B′B′ a pair belonging to the other set; the flow along them being the same. The line CC′ drawn through two of the intersections is one set of the resultant stream lines; and the lines parallel to CC′, drawn through DD′, are two more.

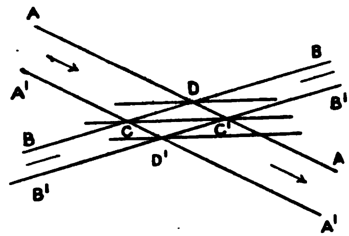

FIG. 4.

For CC′ may be considered a section of the stream AA′, and the velocity perpendicular to CC′ is the flow divided by the area CC′. Similarly, it is a section of the stream BB′, and the velocity across it is flow divided by CC′, which is the same as the flow in the first stream. Since they act in opposite directions, there is no flow across CC′. Similarly, the flow across DD′ due to each stream is the flow divided by the area DD′, and the forces producing the two streams tend to send a double volume per second through DD′. Hence, between D, D′ there are two resultant elementary streams, so that each of the points D, D′ is traversed by one of the resultant stream lines.

The velocity of the stream in AA′ : that in BB′ : that in CC as DC′ : DC : CC′.

The simple types of motion satisfying the condition already laid down are—

(1) Motion in parallel straight lines. Here, $\rho = \infty^{y}$, $dv = 0$, and $dt = 0$, so that the velocity is the same in all the stream tubes, and the distance apart of the stream line is invariable.

(2) Motion in straight lines converging towards, or diverging from, an axis, to which they are all perpendicular.

In this case, considering a layer of uniform thickness, the elementary streams are of the form of wedges, separated from each other by planes radiating from the axis, and making equal angles with each other. The velocity varies inversely as the distance from the centre, and at the centre is infinite.

Thus, there is a *source* by which liquid is perpetually flowing out in all directions, or a *sink* down which it is perpetually flowing, according as the velocity is from or towards the axis. The continued existence of a source or sink would postulate a

continual creation or annihilation of fluid at the point in question.

§ 6. **Combination of Elementary Streams.**—In Fig. 5, A and B represent a source and sink, the streams diverging from A along the streams—which are in the shape of wedges—and the streams diverging towards B along diverging wedges. The resultant

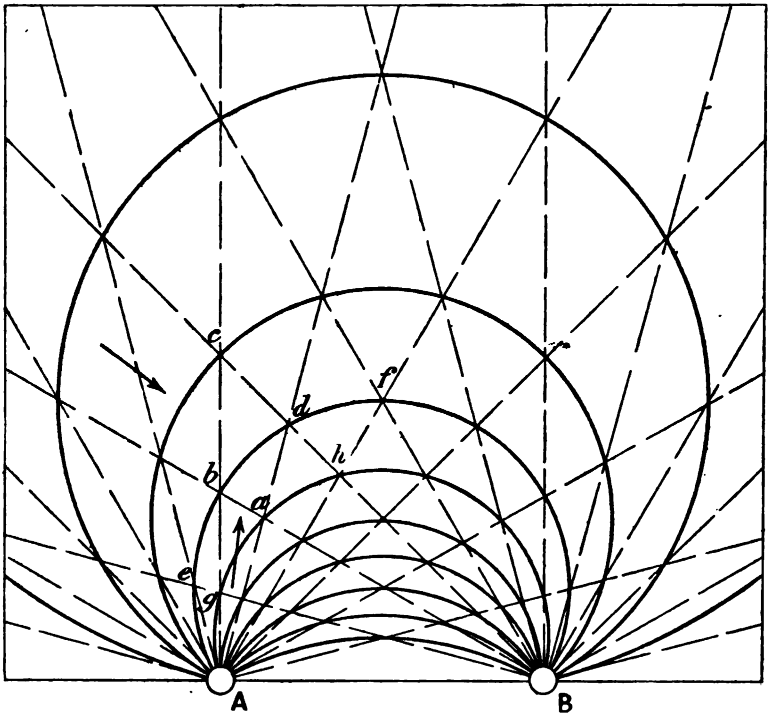

Fig. 5.

stream lines are obtained by drawing curves through the intersections of the network of lines. Thus, considering the space *abcd*, in which direction of the flow is in the direction of the arrows, the stream line *bd* is clearly in the direction of the resultant, and there is no flow across the section *bd*. Thus, two adjacent stream lines are *gah*, *ebdf*. These stream lines all pass through the foci A and B, and, clearly, they are arcs of circles.

The circles will be continuous on the lower side of the axis AB; but, for clearness, only one side has been drawn. The centres of the circles are on a line which bisects AB at right angles. The motion is assumed to take place in an infinite mass of water, and the motion is plane motion.

The rate at which the liquid is supplied to the sink, or discharged from the source, is called the *strength* of the source or sink. If Q be the flow along each wedge per unit of thickness, θ the angle of the wedge, m the strength, then

$$m = \frac{2\pi}{\theta} \cdot Q$$

or, since the flow along each wedge is the same—

$$m \propto \frac{1}{\theta}, \text{ and } v = \frac{m}{2\pi r}.$$

§ **7. Flow past a Plane Oval.**—In Fig. 6, the circular stream lines are continued with the stream lines of a parallel flow. Con-

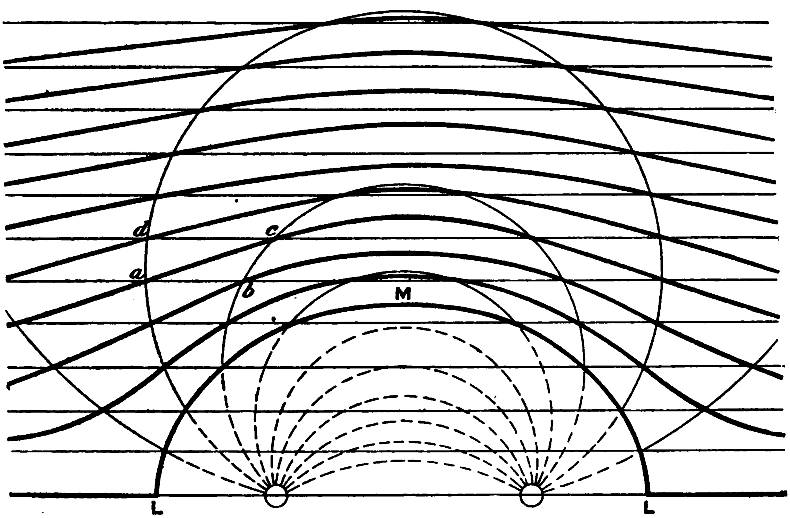

FIG. 6.

sidering a space *abcd*, the resultant stream is along *ac*, and the other streams pass through *b* and *d*. It will be noticed that all

the stream lines are asymptotic to the respective parallel lines. The axis of x, and the stream line LML is what may be called the "solid" stream line. The stream lines inside the solid stream line have not been drawn.

To account for the solid stream line, at an infinite distance, the lines of the parallel flow stream are equidistant apart, and the velocity is v, say. The effect of the source and sink is zero. As a particle of water on the axis x approaches the foci, they come under the influence of the source and sink. If x be the distance of a particle from the point o—midway between the foci—$2a$ be the focal distance, AB and c the velocity in the parallel stream, then the resultant velocity at distance x is

$$c - \frac{m}{2\pi(x-a)} + \frac{m}{2\pi(x+a)} = c - \frac{m}{2\pi} \cdot \frac{2a}{(x^2 - a^2)}.$$

At the apex L, $x = \mathrm{OL} = l$, say; and the resultant velocity is zero, the plane of water at impact being perpendicular to the axis. If the semi-axis be l, at the point L—

$$c = \frac{m}{2\pi} \cdot \frac{2a}{l^2 - a^2}$$

and the resultant volocity is zero. This equation gives a relation between the semi-major axis of the solid stream line, the semi-

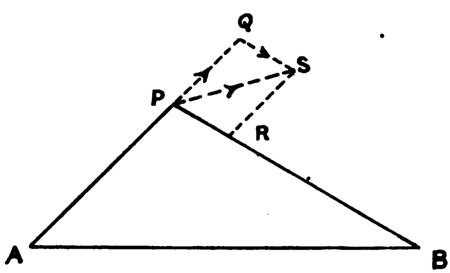

Fig. 7.

focal distance, the strength of the source or sink, and the velocity in parallel flow. Solving, the semi-major axis is

$$l = \sqrt{\frac{ma}{\pi c} + a^2}.$$

Consider a point, P, in a circular stream (Fig. 7). This point is acted by two velocities, a velocity of $\frac{m}{2\pi . \mathrm{AP}}$ in the direction from A to P, and a velocity $\frac{m}{2\pi . \mathrm{BP}}$ in the direction PB. If PQ and PR be set off, in the directions shown, proportional to these velocities, then the triangle PQB is the triangle of velocities, and is similar to the triangle RPB. Hence, if r and r' be the radii, AP, PB, and V the resultant velocity at P, then

$$\mathrm{V} : \frac{m}{2\pi r} = \mathrm{AB} : \mathrm{PB} = 2a : r'$$

whence
$$\mathrm{V} = \frac{m . 2a}{2\pi r r'}.$$

In the limit, when A and B are very near together, but not actually coinciding, $m . 2a$ is finite, and is usually called μ: so that, in the limit—

$$\mathrm{V} = \frac{\mu}{2\pi r^2}.$$

The coefficient μ is defined as the strength of the *double source*.

Again,

$$c = \frac{m}{2\pi} \cdot \frac{2a}{l^2 - a^2} = \frac{\mu}{2\pi(l^2 - a^2)}.$$

When the foci are very near together, this becomes

$$c = \frac{\mu}{2\pi l^2}.$$

Under these conditions, the oval stream lines become circular stream lines, having their centres on a line perpendicular to the axis, and touching at the double source O.

§ 8. **Stream Lines past a Circular Cylinder in Plane Motion.**— Fig. 8 shows the circular stream lines due to a double source. The velocity at any point P is $\frac{\mu}{2\pi r^2}$, OP being the distance r. Consider two circular stream lines, and let the radii be R and R + dR, and take the section at the crown. The velocity is $\frac{\mu}{2\pi(2\mathrm{R}^2)} = \frac{\mu}{8\pi\mathrm{R}^2} = \mathrm{V}$

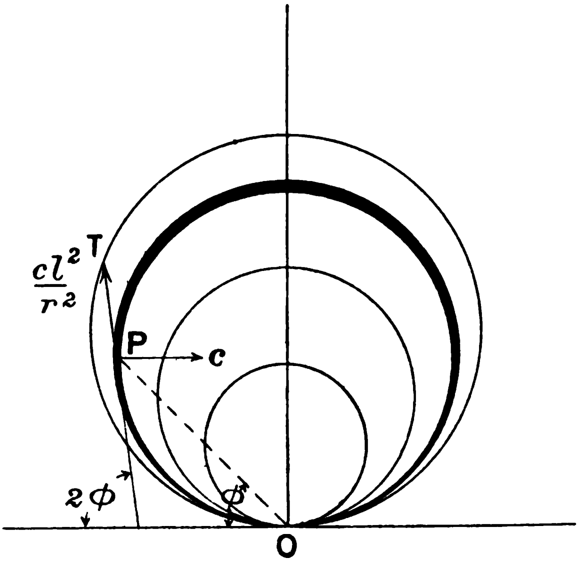

Fig. 8.

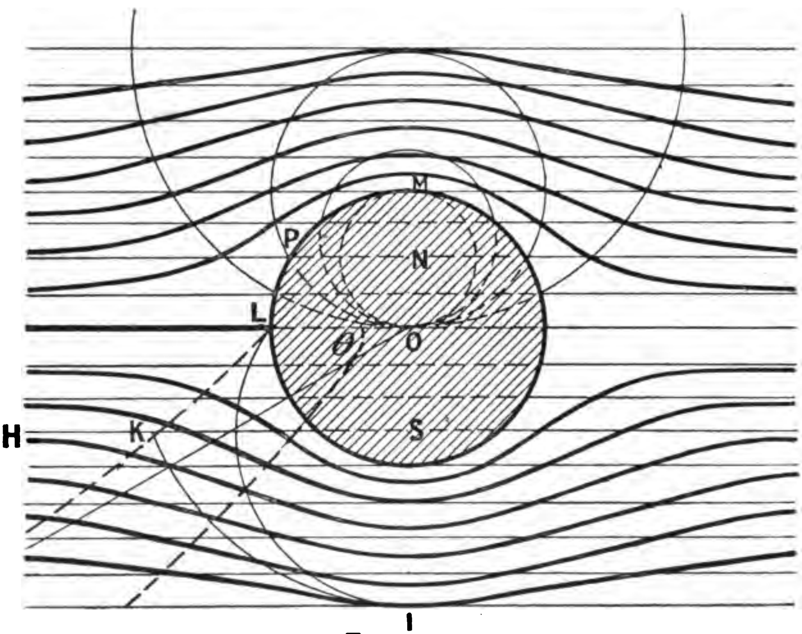

Fig. 9.

and the flow per second is $V \cdot 2dR$. Integrating between R_1 R_2, the total flow is

$$\frac{\mu}{4\pi}\int\frac{dR}{R^2} = \frac{\mu}{4\pi}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

Thus, for tubes of equal flow, the successive radii are in harmonious progression.

The successive radii can be found graphically by the following construction (Fig. 9): Take a circle of centre O and radius l. Divide the vertical radius into a number of equal parts, and draw lines parallel to the axis. Draw circles, touching the axis at O, and passing through the points where these lines cut the circle. The circles thus obtained are the consecutive stream lines of a double source, and if R be the radius of the circle corresponding to P,

$$2R \cdot ON = l^2$$

or

$$\frac{1}{R} \propto ON.$$

Through the points N_1, N_2 draw lines parallel to the axis. These lines represent a parallel stream of equal flow. The circular stream lines must be combined with the parallel stream, as explained in the oval curves in § 7. The flow, obviously, is the flow past a circular cylinder in plane motion. The condition given in § 7 must also be satisfied, namely—

$$c = \frac{\mu}{2\pi l^2}$$

giving a relation between the strength of the stream, and the speed of uniform flow.

§ 9. Equation to the Stream Lines past a Circular Cylinder.—Referring to Fig. 9, the equation to circle defined by N_2, is

$$2R_2 \cdot ON_2 = l^2.$$

If O be the origin, and the axes of x and y are in the direction of flow, and perpendicular to it, the equation becomes

$$x^2 = y(2R - y)$$

or

$$x^2 + y^2 = 2yR = y\frac{l^2}{ON}$$

or

$$\frac{yl^2}{x^2 + y^2} = ON = b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which b is the distance of the asymptote for the stream line corresponding to ON, that is, the distance of the asymptote of the resultant stream line from the axis. The point of intersection of this stream line and the circular lines is such that the ordinate y (at P) is given by

$$y-b=\mathrm{ON}$$

y being given by the first equation. Thus, the equation to the resultant stream line is

$$\frac{yl^2}{x^2+y^2}=y-b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

or

$$b=y\left(1-\frac{l^2}{x^2+y^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which b is a variable parameter defining a particular stream line.

When

$$b=0, \quad y=0,$$

and

$$x^2+y^2=l^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the first equation represents the axis of x, and the second a circle of radius l.

If the cylinder move through still water with a velocity c, the equation to the stream lines relative to still water are the circular stream lines of the double source, and, therefore, the equation

$$\frac{yl^2}{r^2}=\mathrm{ON} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

§ **10. Velocity at a Point in the Resultant Stream Line.**—In § 7 the velocity in the circular stream line at a point is $\frac{cl^2}{r^2}$ acting in the direction of the tangent TP. The velocity in the parallel stream is c. If ϕ be the polar angle, and u, v the component velocities parallel and perpendicular to the axis respectively, then (Fig. 8)

$$u=c-\frac{cl^2}{r^2}\cos 2\phi$$

$$v=\frac{cl^2}{r^2}\sin 2\phi$$

so that, if V be the resultant velocity—

$$V^2 = c^2\left(1 - \frac{2l^2}{r^2}\cos 2\phi + \frac{l^4}{r^4}\right) \quad . \quad . \quad . \quad . \quad (6)$$

in which ϕ and r are connected by the equation—from equation (3)—

$$b = r \sin \phi\left(1 - \frac{l^2}{r^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

At a point on the cylinder, $r = l$, and

$$V^2 = 4c^2 \sin^2 \phi$$
$$V = 2c \sin \phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

This represents the velocity of gliding at a point in the cylinder defined by the angle ϕ.

§ 11. **Change of Pressure along a Stream Line.**—Let p_o be the pressure in the parallel stream, p the pressure at a point in a resultant stream line, the polar co-ordinates of the point being r and ϕ. The excess pressure above the pressure in the parallel flow stream is

$$\frac{p - p_o}{w} = \frac{c^2 - V^2}{2g} = \frac{c^2l^2}{gr^2}\left(\cos 2\phi - \frac{l^2}{2r^2}\right) \quad . \quad . \quad (9)$$

The excess pressure may, therefore, be calculated for any stream line defined by b.

Only the primary stream line formed by the axis and by the circular cylinder will be considered.

For the circular stream line, $l = r$, and the excess pressure is given by

$$\frac{p - p_o}{w} = \frac{c^2}{2g}(\cos 2\phi - \tfrac{1}{2}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When
$$\phi = 0, \quad \frac{p - p_o}{w} = \frac{c^2}{2g}$$

$$\phi = 30°, \quad \frac{p - p_o}{w} = 0$$

$$\phi = 90°, \quad \frac{p - p_o}{w} = -\frac{3}{2}\cdot\frac{c^2}{2g}.$$

Thus, the pressure is greatest at the apex; it becomes zero when $\phi = 30°$, and becomes negative from 30° to 90°, at which it has its greatest negative value. Thus the greater part of the cylinder is subjected to suction.

The axis of x is part of the circular stream line, and, referring to equation (8), ϕ must be zero; so that, substituting x for r, the equation becomes

$$\frac{p - p_o}{w} = \frac{c^2 l^2}{g x^2}\left(1 - \frac{l^2}{2x^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

an equation of the fifth degree.

Fig. 10 shows the curve of excess pressure plotted on the axis of x. The curve is interesting. There is a point of inflection, the position of which could be obtained from equation 11.

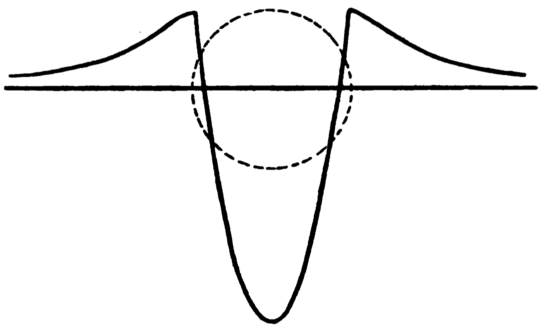

Fig. 10.

§ 12. **Loci of Points of Maximum, Minimum, and Normal Pressure.**—In the stream line past a circular cylinder, or any oval figure, there are three points in the stream line at which the excess pressure is a maximum, normal, and a minimum. At these points the velocity is a minimum, normal, and a maximum. For maximum or minimum—

$$\frac{dV}{dr} = 0$$

Differentiating equation (6)—

$$\sin 2\phi \cdot \frac{d\phi}{dr} + \frac{\cos 2\phi}{r} - \frac{l^2}{r^3} = 0.$$

In a given stream line, b is independent of r, and, therefore, from equation (7)—

$$\frac{d\phi}{dr} = \frac{l^2 + r^2}{r(l^2 - r^2)} \cdot \frac{\sin \phi}{\cos \phi}$$

whence, for points at which the pressure is a maximum—

$$\pm 2 \sin \phi = \frac{l^2 - r^2}{r^2}.$$

The positive sign refers to the lines inside, and is, therefore, irrelevant.

The equation of maximum excess pressure outside the cylinder is

$$l^2 = r^2(1 - 2 \sin \phi) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Where $\phi = 0$, $r = l$. Where $\phi = 30°$, $r = \infty^y$. Thus, the curve has an asymptote passing through the apex L, and inclined at 30° to the axis. It is inclined at 45° to the axis.

At points where the pressure and velocity have the normal values of pressure, that is, the pressure in the parallel flow stream, equation (9) gives

$$\cos 2\phi = \frac{l^2}{2r^2}$$

or

$$x^2 - y^2 = \frac{l^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

a rectangular hyperbola of semi-axis $\frac{l}{\sqrt{2}}$, and asymptote inclined at 45° with the axes.

The points of minimum pressure are obviously along the line $x = 0$, that is, on the line through the origin perpendicular to the axis.

Fig. 9 shows the curves plotted graphically, the stream lines being plotted below the axis for clearness. To draw the curve of minimum velocity, draw the parallel line through H, intersecting the vertical axis in S. Make SI = SL, and, with O as centre and radius OI, describe a circular arc to cut H_3 in K. K is a point on the required locus.

To draw the curve of normal velocity, make $O = \frac{l}{\sqrt{2}}$, and, with S as centre and SL as radius, find the point of intersection K. K is a point on the required locus.

The magnitude of the maximum pressure is obtained from equation (9), and ϕ and r may be obtained by solving equations (6) or (7), or from the diagram. To obtain the excess pressure at any other point of the same stream line, assume some value of ϕ or r, and measure off the corresponding value of r or ϕ; or, equation (7) may be used. By substituting in equation (9), the excess pressure is obtained.

§ 13. **Stream Lines relative to Cylinder.**—In the preceding investigation the cylinder has been fixed, and the water assumed to flow past it. In dealing with ships, the ship moves through still water. To obtain the corresponding equation, a velocity c must be impressed upon the whole system; thus, a double source, moving along an axis, and ejecting fluid into still water. Relative to still water the stream lines will be

$$\frac{yl^2}{r^2} = \text{ON (equation 5)}$$

and, relative to the cylinder—

$$y\left(1 - \frac{l^2}{r^2}\right) = b.$$

Moreover, the absolute velocity of the water relative to still water in the direction of motion is

$$c - u = \frac{cl^2}{r^2} \cos 2\phi$$

and, perpendicular to direction of motion—

$$\frac{cl^2}{r^2} \sin 2\phi.$$

As the cylinder moves on, the particles of water will first move in the direction of the cylinder, then against it, and again, at the stream, with the cylinder. If ϕ lie between $\frac{\pi}{4}$ and $-\frac{\pi}{4}$, the

particles have a component in the direction of the cylinder; if between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$, against it.

The actual path followed by a particle of water, as the cylinder passes, is clearly a looped curve. The arrow shows the direction of motion of the solid body (Fig 11). The dotted line AC is at a distance b from the axis. The particle at A is first pushed forwards,

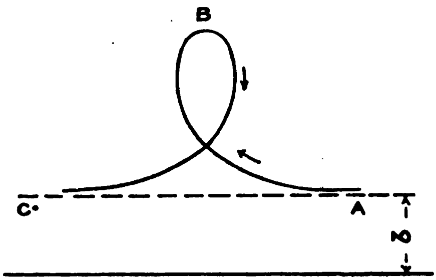

FIG. 11.

then deviated upwards, moves, at some stage, perpendicular to the axis, bends backwards, moves directly against the solid body, as it passes the section of greatest breadth, as shown at B. The particle then turns inwards, and begins to follow the body—the motion being exactly reversed—coming to rest at C in advance of its original position.

The curve ABC is identical with the coiled elastic curve. The property of this curve is that the curvature of the orbit at any point is proportional to the distance of the point from the mid line between the axis and the parallel line AC in Fig. 11.

§ 14. **Virtual Mass.**—In the investigations considered, the body has been assumed to move with uniform velocity through still water; or, the velocity in the parallel fluid stream was assumed uniform. The velocity at every point of the fluid was determinable. The whole system may be considered as a kinematic chain, so that if the velocity of the body changes, the velocity of every particle of water changes in the same ratio. In such a case the whole effect of the fluid may be represented by an addition to the mass of the body. This additional mass, together with the

mass of the body, is called the *virtual mass*. In certain cases, the virtual mass can be readily calculated; but in the general case—ships—it must be determined by experiment. The difficulties which have to be overcome are great, but Mr. William Froude determined the virtual mass of the *Greyhound*, to which reference will be made later.

The virtual mass of a cylinder can be calculated from the principles already considered. The cylinder moves in an infinite mass of fluid with a velocity c, the mass being m; the kinetic energy of the cylinder $\frac{1}{2}mc^2$. For a cylinder, the velocity of any particle, relative to still water, is $\frac{cl^2}{r^2}$. The mass of an element of fluid is $\frac{w}{g}rd\theta dr$. Hence

$$\text{Kinetic energy of fluid} = \int_o^{2\pi}\int_l^{\infty}\frac{1}{2}\cdot\frac{w}{g}\cdot\frac{c^2l^4}{r^3}d\theta dr$$

$$= \frac{\pi wc^2l^4}{g}\left(\frac{1}{2r^2}\right)_{\infty}^{l}$$

$$= \frac{\pi wc^2l^2}{2g}$$

$$= \frac{m'c^2}{2}$$

where m' is the mass of liquid displaced by the cylinder.

Hence, the total kinetic energy

$$= \frac{m + m'}{2}\cdot c^2.$$

If an extraneous force, X, acting on the cylinder, cause the cylinder to be accelerated or retarded, the equation of motion is

$$\frac{d}{dt}\left(\frac{m + m'}{2}\cdot c^2\right) = c\text{X}$$

or

$$(m + m')\frac{dc}{dt} = \text{X}.$$

Writing this in the form

$$m\frac{dc}{dt} = \text{X} - m'\frac{dc}{dt}.$$

The presence of the surrounding fluid is equivalent to a force $-m'\frac{dc}{dt}$ in the direction of motion.[1]

Sphere.—In a sphere, the virtual mass is *half* the mass of the fluid displaced.

§ 15. **Motion past Solids of Revolution.**—So far, the motion considered has been in parallel plane layers, formed by a combination of two elementary modes of motion—namely, motion in parallel streams, and motion from an axis. There is a third simple motion which satisfies the conditions of continuity and irrotationality (§ 3), namely, motion in straight lines converging towards or diverging from a central point. Here the elementary streams are of the forms of cones or pyramids, having their summits at the central point, and of such shapes and sizes as to divide the surface of a sphere, described about that point, into equal areas. The area of a transverse section of an elementary stream varies directly as the square of the distance from the central point, and the velocity of a particle consequently varies inversely as the square of that distance.

In the case of motion in parallel layers, when a source was combined with a sink, the resultant stream lines were circles; and, by combining their resultant with parallel streams, the stream lines represented motion past an oval. By using a double source, the stream lines past a circular cylinder were obtained.

If a current which diverges in all directions from a point, combine with a current converging in all directions to a point, the resultant will consist of a number of wedge-shaped solids; and, if the surfaces be drawn diagonally through the network, the resultant represents the motion of a current which diverges in all directions from one focus, and converges towards the other.[2]

§ 16. **Stream Lines past a Solid of Revolution.**—Take a simple case. Let the parallel stream surfaces be coaxial cylinders, the axis being the line joining the foci. Let the diverging surfaces be coaxial cones, the axes of the cones being coincident with the

[1] Lamb's "Hydrodynamics," p. 86.

[2] These stream lines will be of the same shape as the lines of force of a two-poled magnet.

axis of the cylinders. Under these conditions, the resultant streams represent the flow past a solid of revolution. The motion is assumed to take place in an infinite mass of liquid, and that any particle remains in the same plane.

These cylindrical and conical surfaces have to be so disposed that the flow across each section is the same. In the parallel flow current, the area is the annular space between two concentric circles; in the diverging streams, the surface is part of a sphere described about the foci as centre.

To find the law of spacing of the cylinders, Fig. 12 shows a wedge, its axis coinciding with the axis of the solid. The radii of the cylindrical surfaces are r_1 and r_2, the wedge angle is a. The law of equal flow is satisfied, provided

$$\frac{a}{2}(r_1^2 - r_2^2)v = \text{constant}$$

and v is the velocity in parallel flow.

The wedge, therefore, must be so divided that the squares of successive distances are in arithmetical progression. A geometrical construction is shown in Fig. 13. Let EX be the axis of the solid, and the plane of the paper the plane of the wedge; and suppose that the breadth EA has to be divided into streams of equal flow. With E as centre, and radius EA,

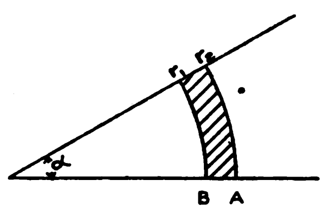

Fig. 12.

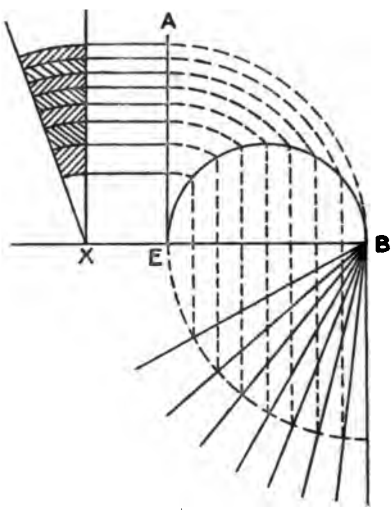

Fig. 13.

describe a quadrant of a circle meeting the axis in B. On BE describe a semicircle. Divide EB into as many equal parts as it is

required to divide EA, and through the points of subdivision draw lines perpendicular to the axis. With E as centre, draw circular arcs passing through the points where these lines cut the circle to meet EA; and through the points of subdivision of EA draw lines parallel to the axis. These lines give the traces of concentric cylinders, which act as stream tubes of flow.

Consider the diverging streams. These are conical surfaces coaxial with the cylindrical surfaces. Fig. 14 shows the trace of the sphere on a plane containing the axis, and a diagrammatic view of a section. O is the centre of the sphere of radius r, and PQ is a small arc, ds The angle POA is θ, and POQ, $d\theta$; and PN, drawn perpendicular to OA, is the radius of the circle passing through P. Thus the element of arc PQ $= rd\theta$, PN $= r \sin \theta$; thus the area of the portion of the sphere between the section through P and Q is

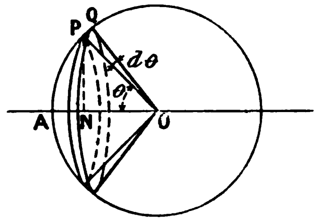

FIG. 14.

$$2\pi r \sin \theta . rd\theta = 2\pi r^2 \sin \theta d\theta.$$

Integrating between limits of θ_1 and θ_2, the area of the spherical portion whose section is PQ is equal to

$$2\pi r^2(\cos \theta_1 - \cos \theta_2).$$

Thus for equal areas the cosines of successive angles of θ are in arithmetical progression. Referring to Fig. 13, draw a quadrant of a circle with centre B and radius BE. Produce the lines already plotted to meet this quadrant as shown, and draw radiating lines from B through the points of intersection. These will be the required stream surfaces for one quadrant round B, and the other quadrants will be similar.

In Fig. 15, A and B are the source and sink. A circle—shown dotted—is described with A as centre, and AB as radius, cutting the axis again in K. This semicircular arc is used to accurately draw the diverging cone sections, which are shown by faint lines. The system is then transferred to B. The resultant system is

shown by the darker lines. These lines are the same as those of a two-pole bar magnet. The cylindrical surfaces are drawn from

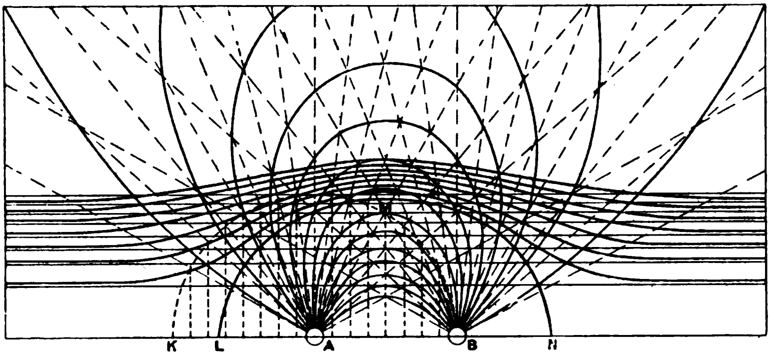

FIG. 15.

Fig. 13. The difference between these stream lines and those of a cylinder (Fig. 9) will be noticed. If the source and sink coincide, the solid of revolution becomes a sphere. It is not proposed, however, to discuss the case any further.

§ **17. Viscous Stream Line Flow.**—In the *Transactions of the Institution of Naval Architects* (1898), Professor Hele-Shaw, F.R.S., exhibited some remarkable results—which were thrown on the screen by a lantern—of viscous flow stream lines. He forced a viscous fluid through flat glass plates very near together, and added at certain points a coloured dye.[1] The conditions were such that the dye was drawn out in a thin line. Professor Hele-Shaw interposed obstacles between the glass; and so obtained the form of the stream lines corresponding to any shaped body. In particular, Professor Hele-Shaw obtained the flow past a circular cylinder. The conditions, however, were not those discussed in § 2. In that case, the motion takes place in an infinite mass of fluid, and the boundaries were straight lines at an infinite distance, and the fluid was perfect. In the experiments, the boundary was straight, but at a finite distance. The solution of this case has

[1] Professor Osborne Reynolds first used coloured dye to show the motion of fluids (§ 18).

been worked out by Professor Lamb, F.R.S.[1] The stream function involved trigonometrical and hyperbolic functions, and the stream lines were plotted by a graphic process. Fig. 16 shows the result

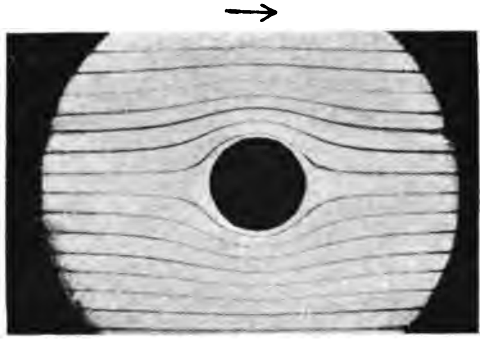

Fig. 16.

obtained by Professor Hele-Shaw, and Fig. 17 shows the stream lines as plotted from Professor Lamb's formula. This result was

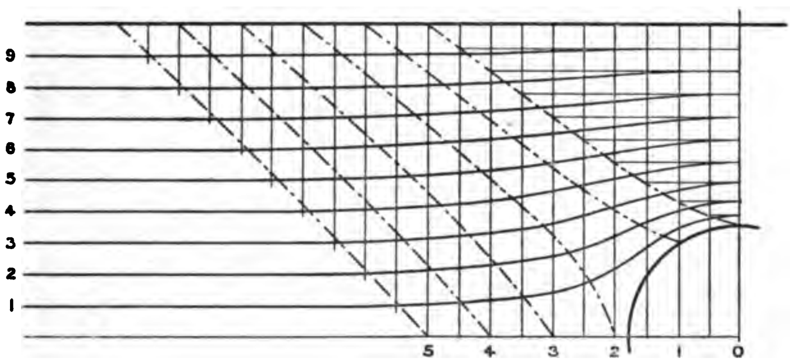

Fig. 17.

unexpected, as the forces operating in the two cases were entirely different. The one was due to inertia, the other to viscous resistance.

The difficulty was removed by Sir George Stokes[2] proving that,

[1] Published in the *Transactions of the Institution of Naval Architects*—in Professor Hele-Shaw's paper—1898.

[2] *British Association Reports*, 1898, Bristol meeting, pp. 143, 144.

under certain conditions, the stream lines of viscous fluid and of a perfect fluid were actually identical in *form*, but in no other respects.

§ 18. **Theory of Viscous Flow.**—In the stream lines so far considered, the fluid is assumed perfect. The lines represented in Figs. 6 and 15 are the stream lines of a perfect fluid, and therefore cannot be reproduced. The laws of motion are those given in §§ 7 and 16, and there is no force tending to push the obstacle along.

All fluids possess the property of viscosity. Consider, first, the unobstructed flow of a fluid between parallel plates, the flow being supposed to take place along lines parallel to a plane. There will be no change of velocity, but there will be a change of pressure on account of viscous resistance.

Consider a horizontal section, and let *abcd* be an element between two planes very near together. This will be called the axis of x, the axis of z being perpendicular to the direction of the plate, the axis of y in a direction parallel to the plates; u represents the velocity on the face ab, and $u + du$ the velocity on the face cd (Fig. 18).

Now, the viscous forces over ab, cd produce the same effect as two tangential forces. The law of viscous force is that the tan-

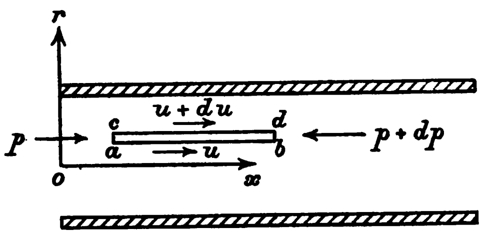

Fig. 18.

gential force between two adjacent faces is proportional to the increment of velocity divided by the distance between faces, and is also proportional to the area. Thus

$$\text{viscous force} \propto \frac{du}{dz} = \mu\frac{du}{dz}$$

where μ is the *coefficient of viscosity*. Thus

the viscous force over the face $ab = \mu\frac{du}{dz}\,.\,dxdy$

$$\text{,,} \qquad \text{,,} \qquad cd = \left\{\mu\frac{du}{dz} + \frac{d}{dz}\left(\mu\frac{du}{dz}\right)dz\right\}dxdy.$$

The difference, or the force tending to drag the element along, is

$$\mu\frac{d^2u}{dz^2}dxdydz.$$

If p, $p + dp$ be the intensities of pressure over ac and bd, then

$$dp \times dydz = \mu\frac{d^2u}{dz^2}dxdydz$$

or

$$\frac{dp}{dx} = \mu\frac{d^2u}{dz^2}.$$

In this equation u is a function of x. There can be no variation of pressure in the direction of y and z. Thus, if p_1 and p_2 be the two pressures at the end faces of a strip of length l—

$$\frac{d^2u}{dz^2} = \frac{p_2 - p_1}{\mu l}$$

whence

$$u = \mathrm{A} + \mathrm{B}z + \frac{z^2}{2\mu l}(p_2 - p_1).$$

At the boundaries—$2h$ being the width between the plates—there is no slipping; therefore when $u = 0$, $z = \pm h$, and, therefore—

$$u = \frac{h^2 - z^2}{2\mu l}(p_1 - p_2).$$

The maximum velocity is at the centre, and is

$$\frac{h^2}{2\mu l}(p_1 - p_2).$$

The curve representing the variation of velocity is a parabola, and the mean velocity is two-thirds the maximum. If u_o is the mean velocity, then at any point

$$u = \frac{3}{2}u_o\left(1 - \frac{z^2}{h^2}\right).$$

The flow is $\int_{-h}^{+h} u dz$, and is equal to

$$\frac{2h^3}{3\mu} \cdot \frac{p_1 - p_2}{l}.$$

§ 19. Application to Professor Hele-Shaw's Experiment.—Imagine flow to take place between two sheets of plate glass placed very near together—say, almost the thickness of a piece of paper apart. Imagine, also, that the fluid is very viscous—say, glycerine—and that the velocity is very small. If, between two sheets of glass, an obstacle be placed, the viscous stream line is sharply drawn out and the motion is stable. It has been pointed out that these stream lines are identical in form with those of a perfect flow.

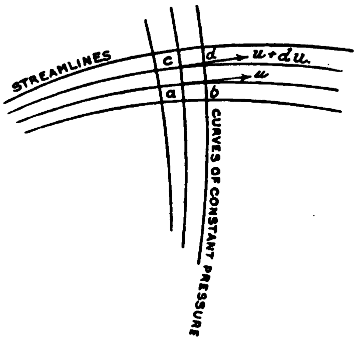

FIG. 19.

Consider (Fig. 19) a network of the stream lines. At a point, let s, n, and z refer to the direction of motion, the normal to the stream line, and the direction perpendicular to the plates. Then since the motion is very small, there is no change of pressure in the direction of the normal; nor in the direction of the axis. Thus the equations of motion are

$$\frac{dp}{ds} = \mu\frac{d^2u}{dz^2} \quad (\S\ 18)$$

$$\frac{dp}{dn} = 0$$

$$\frac{dp}{dz} = 0$$

Thus the line of equal pressure cuts the stream lines orthogonally.

Again (§ 18), the velocity at distance z from the central plane

$$u = \frac{3}{2}u_o\left(1 - \frac{z^2}{h^2}\right)$$

in which u_o is the average velocity across the section. Hence—

$$\frac{dp}{ds} = -\frac{3\mu}{h^2}u_o.$$

Fig. 19 shows the stream lines and lines of constant pressure. Consider an element *abcd*.[1] Then, since *ac*, *bd*, produced, meet at the centre of curvature O of the stream line,

$$\frac{u_o + du_o}{u_o} = \frac{cd}{ab} = \frac{\rho}{\rho + t}$$

in which u_o, $u_o + du_o$ are the velocities at c and d, and ρ the radius of curvature OC, and t the thickness of tube; thus—

$$\rho du_o + u_o t = 0$$

or

$$\frac{u_o}{\rho} + \frac{du_o}{t} = 0.$$

The equation of continuity is

$$u_o t = \text{constant}.$$

These two equations are identical with those obtained for a perfect fluid (§ 2).

Thus, the stream lines of a perfect fluid are identical in form with those of a viscous fluid, provided a very viscous fluid is used, the motion very slow, and the width between the plates small. In other respects, the systems are dissimilar.

[1] See *Transactions of the Institution of Naval Architects*, 1900, pp. 225–228.

CHAPTER II

WAVES

§ **20. General Considerations.**—In considering waves, it is important to notice that water motion is different from wave motion. When on board ship, large waves, which are rushing towards the ship with a velocity of many miles per hour, do not carry the ship with them, but pass under the bottom without appreciably moving it out of its course. Similarly, when waves are approaching the shore, a floating object immersed in the water near its surface is not carried forward towards the shore with the rapidity of the wave, but is left in the same place after the wave has passed them. If the tide is ebbing, the object recedes out to sea. Thus the motion of a wave is different from the motion of the water in which it moves; the water may move in one direction and the wave in another.

The things to consider are: (1) The *wave motion*—its range of transmission over the surface of the water, the velocity of transmission, the form of the elevation, its amplitude, breadth, height, volume, period; and, also, the path which each water particle describes during the wave transit—the form of that path, the vertical and horizontal ranges, and the variation of path with depth.

Waves may be subdivided into three classes—

(1) Waves of translation.

(2) Oscillating waves, or ocean waves.

(3) Capillary waves, or ripples.

§ **21. Types of Waves.**—In the *wave of translation* the wave is wholly raised above, or wholly depressed below, the general surface of the fluid. In the former case it is called a

positive wave; in the latter case, a *negative* wave. Moreover, in a perfect wave of this type, there is a single hump or hollow; so that the wave, which one so seldom sees, has the appearance of a hump or hollow running on the surface of still water. The great tidal wave is of such a type, but as it extends from the Thames to Aberdeen, it is impossible for the eye to take in its forms or dimensions. The wave of translation is frequently observed on the River Severn, where it causes a severe wash on the banks. The wave may, however, be generated in a tank. In such a wave, not only does the wave form progress, but each particle of water is actually translated a (short) distance in the direction of motion.

Oscillating waves, which are the most familiar, never occur singly, but always in groups. They are raised partly above and depressed partly below the surface of still water. As the wave form passes, each particle of water describes a closed path, and so resumes its former position. It is not carried permanently forward.

Capillary waves, or ripples, are very similar to oscillating waves, with this difference. In oscillating waves the motive force is almost entirely due to gravity, the effect of capillary action or surface tension being small in comparison; in ripples, the motive force is almost entirely due to tension, the effect of gravity being small in comparison to it. Ripples can be generated by drawing a fishing-line through the water, or by throwing a stone in a pond.

Example of the three types.—The three kinds may be illustrated by considering the action of the wind upon a perfectly still sheet of water. For a velocity of less than $\frac{1}{2}$ mile an hour (9·1 inches per second), the smoothness of the water is undisturbed; if the velocity be 1 mile, say, the water surface becomes covered with delicate tracery, which disappears the moment the wind drops. These are ripples. If the wind increase to, say, 2 miles an hour, the waves become larger, and cover the water to a considerable extent with great regularity, and which do not disappear immediately the wind ceases. These are the ordinary oscillating waves, which continue enlarging their dimensions as the wind increases. It is not until the waves of the oscillating type encounter a shallow shelving beach that they present any of the phenomena of the wave of translation. After breaking on the margin of the shore,

they continue to roll along in shallow water towards the beach, and become transferred into waves of translation, finally breaking on the beach.

Wave of Translation

§ **22. Genesis.**—A wave of translation is due to a local disturbance. Mr. Scott Russell[1] was the first to make experiments on waves.

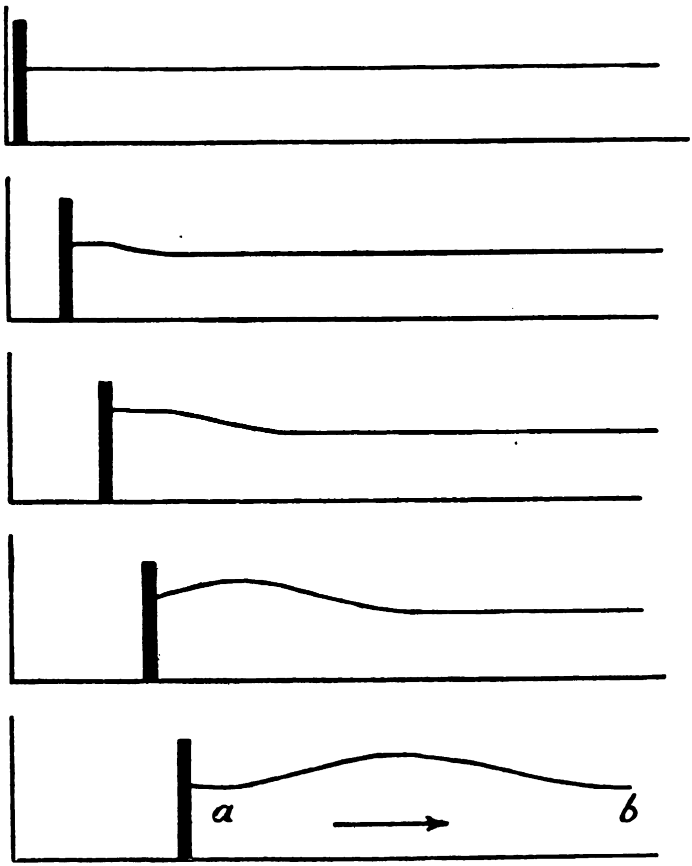

Fig. 20.

The apparatus consisted of a long, narrow channel 12 inches

[1] *British Association Reports*, 1844.

wide, 8 inches deep, 30 feet long, filled with water to a depth of 4 inches. A flat board (or glass), fitting the tank and capable of being moved along so as to push the water in front without allowing it to escape at the sides, or capable of sliding up and down so as to act as a dam.

The wave may be generated in three ways—

(1) *By impulsion or force applied horizontally.*—The plate is moved forward with an increasing velocity and then gradually brought to rest. The height to which the water is heaped up increases with the speed, and when the plate is brought to rest again, the level of the water in front of the plate is then the original level (Fig. 20). But the wave, the volume of which is

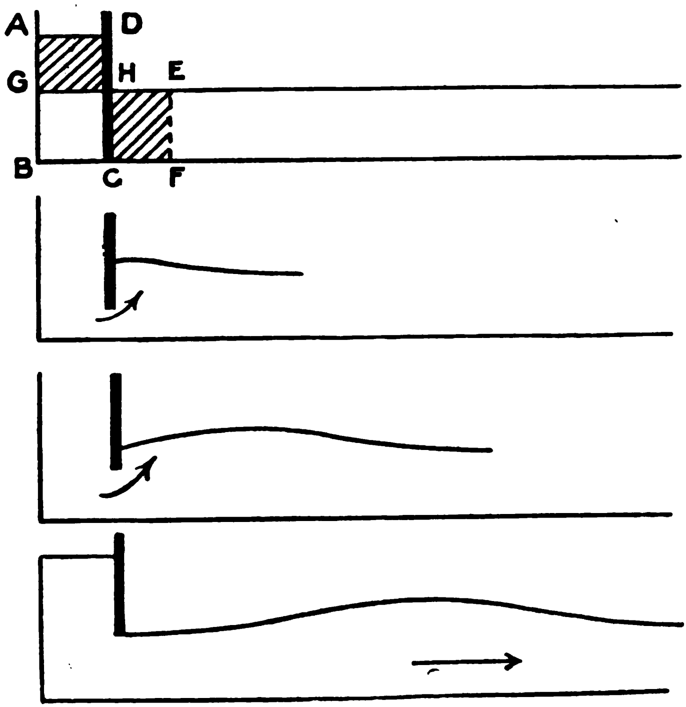

Fig. 21.

equal to the volume displaced, moves on. The length of the

wave in the direction of translation is represented by *ab*, and is called the amplitude.

(2) *By a column of fluid.*—This method is useful when it is necessary to measure forces or volumes in wave genesis. If the quantity of water displaced in this case, as in case (1), be of the same amount, the wave produced is precisely the same form and size. The volume of the water in the wave form is exactly the same as the volume in the reservoir, but it is not the *same* water. Fig. 21 shows the method of generating the wave. The water in the column ABCD was tinged with dye, and the line of colour when the water reached the level of the water in the tank was EF, in which the volume HEFG or ADHG is the volume of genesis, that is, the volume of the wave.

(3) *By protrusion of a solid.*—This is a useful method when large waves have to be generated (Fig. 22).

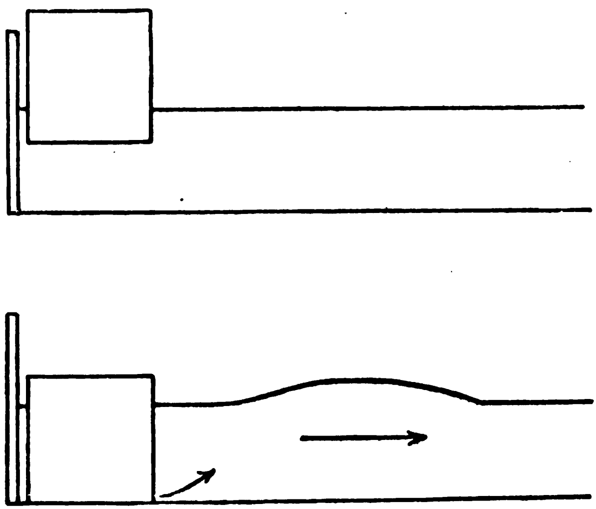

FIG. 22.

§ **23. Types of Waves.**—The wave of translation is thus a means of transmitting energy. The wave, progressing along, might, at the other end, be enclosed by a shutter. When the wave reaches the end of the tank, it is reflected back, and it might be reflected many times from end to end with appreciably

diminishing height. Thus it appears that there is little dissipation of energy in a wave of translation.

During Mr. Scott Russell's careful experiments on the resistance of barges in narrow canals—the resistance being measured by a dynamometrical apparatus in the tow-rope—a spirited horse, alarmed at something, started off at a rapid rate. Before, the water in the rear of the boat was very agitated, and there was a great surge on the bank. But when a certain speed was reached, the surging disappeared, and the barge practically was on a single wave; and the resistance did not increase. This showed the great difference between oscillating waves and waves of translation.

Residuary waves.—If the volume of the column of generation considerably exceed the volume of a wave equal in height to that of water, the water will assume the form notwithstanding, and will press forward with its usual height and form; it will free itself of the redundant water by which it is accompanied, leaving it behind; and this residuary wave will follow after it, only with a less velocity, so that although the two waves at first unite in a compound wave, they afterwards separate and get progressively further apart the further they travel. The residuary wave in this case is a *positive* wave (Fig. 23). On the

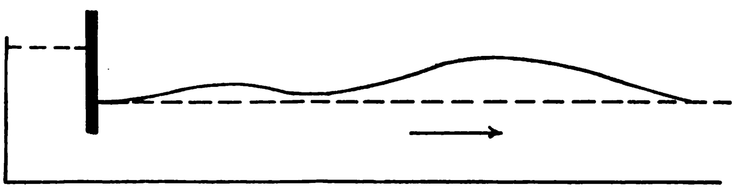

FIG. 23.

contrary, when the height of the generating column is great compared to its breadth, the tendency is to generate a wave of height greater than that corresponding to the volume of generation when a "suction wave" is generated, the extra volume being obtained from the channel itself (Fig. 24). The residuary wave is called a *negative* wave.

It is of importance to notice that these residuary waves are not companion waves. They accompany the genesis of the wave,

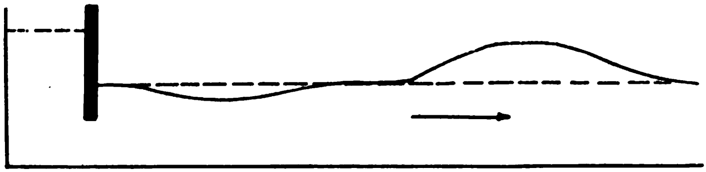

FIG. 24.

but do not attend the transmission, as they are rapidly left behind by the great primary solitary wave. Moreover, the negative wave is very unstable.

§ 24. **Experimental Results.**—(1) *Degradation of a wave.*—A wave of translation can travel considerable distance with little degradation. The longevity depends on the depth and width of channel, and also the height of wave—increasing as these increase. The degradation of height is observed to proceed more rapidly in proportion as the channel is narrow, shallow, or irregular, and according as the sides are rough. The degradation takes place by the particles near the sides and bottom being retarded by friction, and so gradually reducing the volume of the wave. The volume is thus gradually diffused over a large extent along its path, and so finally disappears.

(2) *Velocity of propagation.*—The velocity of propagation depends on the depth of water and the height of wave, the velocity being given by the equation

$$v = \sqrt{g(h + k)}$$

in which h is the depth of water, and k the height of wave.[1]

A high wave in shallow water may move faster than a shallow wave in deeper water.

This formula has been accurately verified by observing a wave of initial height of 1·34 inch every 40 feet for a distance of 1160

[1] Experiment shows that if $k > h$, on reaching this value, or before, the wave breaks at crest.

feet, when the height was 0·08 inch, the corresponding velocities being observed and calculated. The depth of water in repose was 5·1 inches, the breadth of channel 2 inches, and the genesis column 4·45 cubic inches.

(3) *The length of the wave.*—The length of the wave, unlike the velocity, does not increase with the height of the wave, but appears to decrease with it. In small heights the length,[1] λ, is

$$\lambda = 2\pi h.$$

When the height of the wave is appreciable, the length is

$$\lambda = 2\pi h - a$$

where a has still to be determined.

(4) *Form of the wave.*—The form of the wave surface, for small waves, is sensibly a curve of versed sines, the horizontal ordinates of which vary as the arc of a circle of radius h, and the vertical ordinates are the versines of a circle of radius $\frac{k}{2}$, where k is the height of the wave. Thus—

$$x = h\theta$$

$$y = \frac{1}{2}k \text{ versin } \theta = \frac{k}{2}(1 - \cos\theta).$$

(5) *Actual motion of each particle of water.*—As the wave form passes, each particle of water is lifted up vertically and, in addition, carried forward through a short distance, the actual path being, as near as possible, a semi-ellipse, with the major axis horizontal—the height being equal to the semi-minor axis. The motion of particles at different depths were noticed by having small globules of wax connected to very slender stems so as to float at any depth. It was found that all the particles in a transverse plane before the wave passes remain in the same plane after the water has passed, so that the range of horizontal translation is the same at all depths. The range of horizontal translation was equal to the volume of the wave divided by the waterway of the channel. The vertical displacements are proportional to the

[1] The term *length* is "a relative term." It represents the distance between the section where the height of the still-water level is inappreciable.

distance from the bottom. At a height h' from the bottom, the vertical range is $\frac{h'}{h} . k$.

The wave form is shown in Fig. 25. The curve *abc* is a curve of versed sines on the assumption of no horizontal displacement;

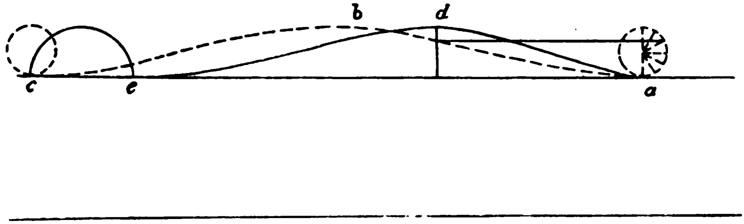

FIG. 25.

the curve *ade*, which represents the true profile, shows the effects of the uniform horizontal displacement. The horizontal displacement, a, say, is given by

$$a = \frac{\text{volume of wave}}{\text{area of waterway of channel}} = \frac{\frac{k}{2} \times 2\pi hb}{bh}$$
$$= k, \text{ for waves of small height.}$$

For large waves, the mean height will be less than $\frac{k}{2}$, so that a varies between πk and $2k$.

(6) *Method of transmission.*—The water in the channel may be subdivided into a number of equal columns, which, since the horizontal motion is the same at all depths, will remain independent. As the front of the wave approaches the column a (Fig. 26), a

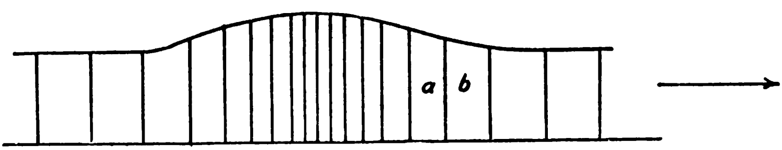

FIG. 26.

pressure is exercised on the posterior side of a which causes that plane to be pushed forward, and so cause an increase in the height of column a. This causes a pressure on the posterior plane of b,

which is likewise pushed forward, causing a rise in b; but, simultaneously, the posterior plane of a is subjected to a still greater pressure, owing to the advance of the wave, which causes an additional rise of the water in column a, and, in turn, a further rise in b; and so on. The motive power thus stored during the anterior half of the wave is restored in the latter half of the wave length—the column raised to its greatest height presses equally on both its posterior and anterior faces. On the anterior surface it presses forward the anterior column, tending to sustain its velocity and maintain its height; on the posterior column, its pressure tends to oppose the progress and retard the velocity of the fluid, and thus retarding the posterior and accelerating surface, widens the space between its bounding planes until it reposes more on the original level. At each process, a certain volume of water is displaced to form the wave, and the net result is that a column equal to the volume of the wave is displaced in the direction of transmission.

(7) *Negative wave.*—The above remarks refer principally to the positive wave of translation. They apply equally well to a negative wave of translation. The negative wave is formed under precisely the reverse conditions to the positive wave. When a solid is drawn from the water at one extremity of the tank, a cavity is created, and this cavity is propagated along the surface of the water. The velocity of the negative wave in a shallow channel is nearly that which is due to the depth calculated from the lowest point in the wave, but in longer waves it is sensibly less than that velocity. The horizontal translation in the negative wave presents considerable resemblance to the corresponding phenomena in the positive wave. All the particles of water in a given vertical plane move simultaneously with equal velocities backwards in the opposite direction to the transmission, and repose in their new places at the end of the translation, with this modification, that the negative wave always gives rise to a train of oscillating waves, which disturb the state of repose near the surface, but which do not sensibly agitate the particles considerably removed from the surface.

§ **25. Theory of the Wave of Translation.**—It has been pointed

out that when a wave of translation is propagated through water, the horizontal displacement of every point is the same.

In a positive wave, the velocity of a particle has its maximum forward velocity at the crest of the wave. Let V_o be the velocity, in feet per second, of the progressive wave (Fig. 27). Impress

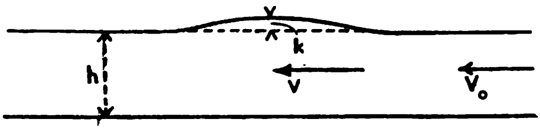

FIG. 27.

upon the whole system a velocity V_o, so that the crest remains stationary, and the water is imagined to flow along lines—the boundary lines being the profile of the surface and the bottom of the tank. This will not in any way alter the distribution of pressure or motion in the wave system. The state of affairs is represented by Fig. 27, in which V_o is the velocity in the normal stream, and V the velocity at the crest of the wave—which is the same for all depths. Such a wave is called a *stationary wave.*

At the crest section and normal sections, the flow is the same, and therefore

$$V(h + k) = V_o h.$$

Also, using Bernouilli's equation, and taking the surface stream line, along which the pressure is constant—

$$(h + k) + \frac{V^2}{2g} = h + \frac{V_o^2}{2g}$$

$$\therefore \frac{V_o^2}{2g}\left\{1 - \frac{h^2}{(h + k)^2}\right\} = k$$

or

$$V_o^2 = 2g\frac{(h + k)^2}{2h + k}$$

$$= 2g(h + k) \cdot \frac{1}{1 + \dfrac{h}{h + k}}$$

Usually k is small compared to h, so that, very approximately—

$$V_o^2 = g(h + k).$$

In waves of small height, such as the tidal wave—

$$V_o^2 = gh.$$

The absolute velocity of each particle of water is

$$V_o - V = V_o - V_o \frac{h}{h+k}$$

$$= V_o \cdot \frac{k}{h+k}.$$

Oscillating Waves in Deep Water

An inspection of the movements of floating bodies show that the particles of water move backwards and forwards through a horizontal distance, which in some cases is equal to, and in others greater than, the extent of their vertical motion; and that these movements are combined in such a manner that each particle of water revolves in a vertical plane in a circle, and, in shallow water, in the shape of a flattened oval.

§ **26. Trochoidal Waves.**[1]—The most important oscillating wave is the *trochoidal wave*, which practically represents *ocean waves.* In such a wave, all the particles of water move in circles, and there is no motion of translation. Only the wave form is propagated on. Moreover, each particle of water, at whatever depth, is assumed to move about some centre with a uniform circumferential motion; but the radii of those circular orbits decreases with depth.

The way in which the wave form is propagated over the surface of water is shown in Fig. 28. The dotted circles represent

[1] See Rankine's "Collected Papers," pp. 481–490.

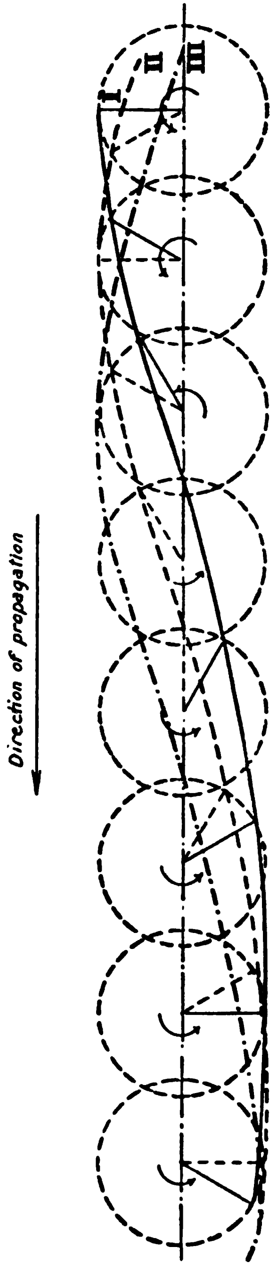

Fig. 28.

the circular orbit of successive surface particles—half a wave length being shown. The direction of rotation is supposed to be counter-clockwise, each particle moving about its proper centre. The full curve I shows the wave form when the crest is at the section. It has been drawn to the form of a trochoid. When each radius has turned through an angle of 30°, the wave form is represented by curve II; and when each radius has turned through 60°, the wave form is represented by curve III; and so on. Thus, the particles move in the direction of propagation at the crest, and in the opposite direction at the trough.

The wave form may be brought to rest by impressing upon the whole system a velocity equal and opposite to that of propagation. The wave then becomes a stationary wave. Fig. 29 shows a progressive wave; Fig. 30 shows a stationary, the direction of rotation being counter-clockwise, and that of the centre towards the right. The motion is a compound motion—a motion of rotation about a centre, and a motion of rolling along a straight line.

In thus reducing it to steady motion, the actual form of the surface stream line is a trochoid. The velocity of a particle at the crest is retarded, and at the trough accelerated.

§ 27. **Velocity of Propagation.**—

Let V be the velocity of propagation of the wave;

O the angular velocity of particles in the orbits;

r the radius of the orbit circle, so that $2r$ is the height of the wave.

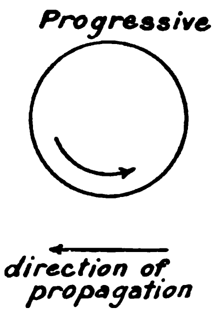

FIG. 29.

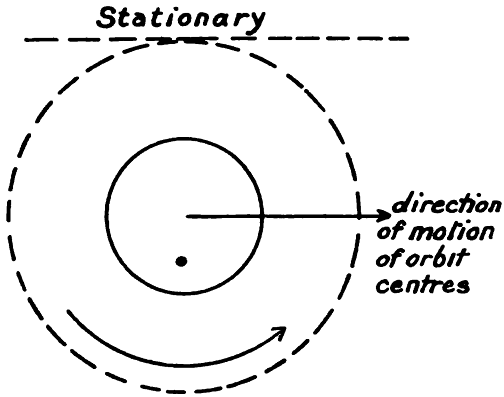

FIG. 30.

Any particle P is rotating about a centre O with the velocity ωr, and is moving forwards with a velocity ωR. These velocities

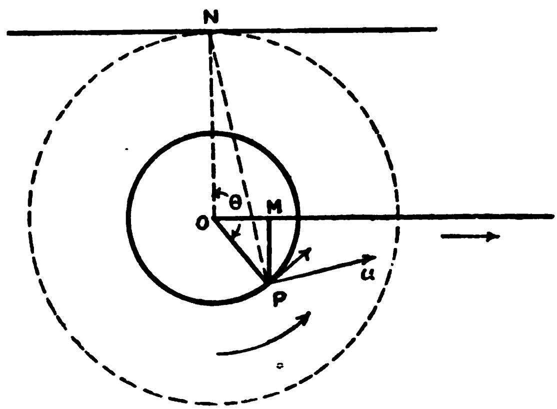

FIG. 31.

may be represented by OP and ON respectively, those lines being perpendicular to the motions they represent (Fig. 31). The resultant velocity will then be represented by PN, and the direction

of motion at P will be perpendicular to PN. Thus, PN is the normal to the surface at P, and the velocity of P is $\omega \, . \, \mathrm{PN}$.

This might have been seen otherwise from the fact that the circle is turning about N as instantaneous centre with angular velocity ω, so that the direction of motion is perpendicular to PN and its velocity is $\omega \, . \, \mathrm{PN}$.

The form so far assumed has been a trochoidal form. This must be justified. Certain relations must exist between the different quantities as will ensure the trochoidal surface being a surface of constant pressure, since this surface is subject to atmospheric pressure.

§ 28. **Velocity at any Point in the Surface.**—Referring to Fig. 31, let u_o be the velocity at the crest, and u the velocity at any point P. Then, if the pressure and the point P and at the crest are the same—

$$\frac{u^2}{2g} - \mathrm{PM} = \frac{u_o^2}{2g} + r$$

the datum line being at the line of centres, and the proper sign, attached to PM. Hence—

$$\frac{u^2}{2g} + r\cos\theta = \frac{u_o^2}{2g} + r$$

θ being the angle NOP, and $\cos\theta$ carrying the proper sign; this, by substitution, becomes

$$\frac{\omega^2 \, . \, \mathrm{PN}^2}{2g} - \frac{w^2(\mathrm{R} - r)^2}{2g} = r(1 - \cos\theta)$$

$$\text{or } \frac{\omega^2}{2g}\{\mathrm{R}^2 + r^2 - 2\mathrm{R}r\cos\theta - (\mathrm{R}^2 - 2\mathrm{R}r + r^2)\} = r(1 - \cos\theta)$$

$$\therefore \frac{\omega^2}{2g} \cdot 2\mathrm{R} = 1$$

and

$$\omega^2 = \frac{g}{\mathrm{R}}.$$

Thus, the trochoidal form of wave at every point on the surface satisfies the condition of constant pressure, provided

$$\omega^2 = \frac{g}{\mathrm{R}}$$

in which case

$$\mathrm{V}^2 = g\mathrm{R}.$$

The length of wave $= 2\pi R = l = \dfrac{2\pi V^2}{g}$, and the height $= 2r$, and the relation between length and speed is independent of height.

Another way of proving the result (Fig. 32).

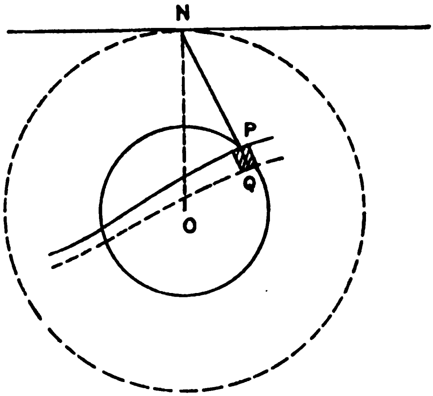

FIG. 32.

In the ordinary progressive wave, P describes a circle about the centre O. The angular velocity of OP is ω; hence the forces acting on a particle of mass m at P are $m\omega^2r$ radially outwards, and mg acting vertically downwards. If NP be normal to the surface at P, then OPN is a triangle of forces, and

$$\frac{m\omega^2 r}{mg} = \frac{r}{R}$$

or

$$\omega^2 = \frac{g}{R}.$$

The resultant force is $m\omega^2$PN.

The apparent weight, therefore, of a small floating body varies from $m\omega^2(R - r)$ at the crest to $m\omega^2(R + r)$ at the trough.

§ 29. **Variation of Pressure with Depth.**—The above only applies to the surface stream lines along which the pressure is constant. The sub-surface stream lines will be assumed to be trochoidal curves of the same length, obviously, as the surface stream line, but clearly different in orbit radii. If the velocity of these

sub-surface waves have the same velocity as the surface wave—they must be surfaces of constant pressure.

Fig. 32 represents the surface stream line, and the one immediately adjacent to it, so that NP is the normal to the surface. Considering the stream tube shown, the flow across every section is the same, so that u being the velocity of the point P in the stationary wave—

$$\begin{aligned} \mathrm{PQ} \,.\, u &= \text{constant}, \\ \mathrm{PN} \,.\, \mathrm{PQ} &= \text{constant}, \\ \text{or } nt &= \text{constant}, \end{aligned}$$

in which n is the normal NP, and t the thickness PQ.

The variation of pressure across the stream lines is also obtained by considering the stationary wave. The element PQ rotates, at the instant, about the instantaneous centre of rotation, N. The resultant force on the element acts along NA, and is clearly

$$w\frac{\mathrm{PQ}}{g}\omega^2 \,.\, \mathrm{PN}$$

taking a unit area of cross section of the element. Thus—

$$dp = w\frac{nt}{\mathrm{R}}.$$

Since nt is constant, and R is the same of all stream lines, dp is constant; and thus *all* sub-surface stream lines are lines of constant pressure.

This may be proved also as follows:—Considering any two adjacent stream lines, let dy_o be their distance apart in still water. Then in each wave length the volume of water enclosed is $2\pi\mathrm{R} \,.\, dy_o$, which must equal

$$\int_o^{2\pi} \mathrm{PQ} \,.\, \mathrm{PN} \,.\, d\theta = 2\pi nt$$

so that $$nt = \mathrm{R}dy_o.$$

But $$dp = \frac{w}{\mathrm{R}}nt = wdy_o.$$

Thus by integration, the pressure in any layer of a trochoidal stream is the same as the pressure at the corresponding depth in still water.

§ **30. Variation of Orbit Radius with Depth.**—In Fig. 33, let two adjacent orbit centres be represented by O, O′, the lines on which the circles roll being NZ, N′Z′ (one drawn full, the other dotted). Let P, P′ be two points which were in the crest

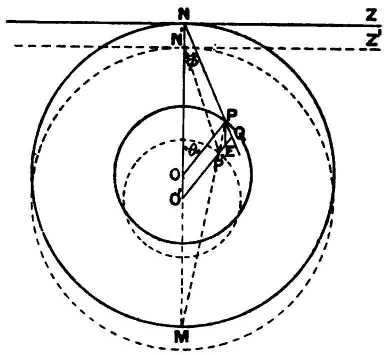

FIG. 33.

section together, so that OP, O′P′ rotate at the same rate and are always parallel.

Let PQ be the common normal to the stream line at P.

Let OP $= r$; O′P′ $= r' = (r + dr)$

NP $= n$; N′P′ $= n' = (n + dn)$

OO′ $= \delta y$; ϕ the angle between ON and NP.

Then $$nt = \text{constant} \qquad (1)$$

Project NN′P′ along NQ; then

$$dy \cos\phi + n' = n + t$$

or $$t = dn + dy \cos\phi$$

whence $$ndn + ndy \cos\phi = \text{constant} \qquad (2)$$

Again, project NPO vertically, giving

$$n \cos\phi = \mathrm{R} - r\cos\theta \qquad (3)$$

Also $$n'^2 = \mathrm{R}^3 + r'^2 - 2\mathrm{R}r' \cos\theta$$

$$n^2 = \mathrm{R}^2 + r^2 - 2\mathrm{R}r \cos\theta$$

whence $$ndn = rdr - \mathrm{R}dr \cos\theta \qquad (4)$$

Substituting (3) and (4) in (2),—

$$(rdr + \mathrm{R}dy) - \cos\theta\,(\mathrm{R}dr + rdy) = \text{constant} \quad . \quad . \quad (5)$$

The results are independent of θ

hence $$\mathrm{R}dr + rdy = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and *also* $$rdr + \mathrm{R}dy = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In (6) $$dy = -\mathrm{R}\frac{dr}{r}$$

$$\therefore [y] = [-\mathrm{R}\log_\epsilon r].$$

In this equation y is measured below the line of orbit centres of the surface stream lines. If O be the centre of the path described by the surface particles, r_o the radius of the path, then when

$$y = 0, \quad r = r_o$$

and $$-y = -\mathrm{R}\log_\epsilon\frac{r_o}{r}$$

$$y = -\mathrm{R}\log_\epsilon\frac{r}{r_o}$$

and, therefore, $$r = r_o\epsilon^{-\frac{y}{\mathrm{R}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus, as the depth below the line of surface orbit centres increases in arithmetical progression, the radius of the orbit circles decreases in geometrical progression.

A geometrical construction may be deduced by using the result in equation (6). Thus, in Fig. 33, knowing OP, to find OP′ draw O′Q parallel to OP, and PE parallel to ON, to meet OQ in E. Make EP′ equal to EQ. Then O′P′ is the radius required.

$$\mathrm{EQ} = dy\,.\frac{r}{\mathrm{R}} = -dr.$$

Hence $\mathrm{P'E} = -dr, \quad \mathrm{O'E} = r, \therefore \mathrm{O'P'} = r' = (r + dr).$

§ 31. **Wave Columns and Surfaces of Constant Pressure.**—Referring to Fig. 33, all points, such as P, P′, which were originally in the crest plane, lie along a curve PP′, which is asymptotic to NO. Such a curve is called a *wave column*. At the crest plane, the column is vertical, but as the wave form passes, it bends gradually over and back again. The tangent to such a wave column at any

point P is PP′, and this intersects NO in M. The point of intersection is shown from the geometry of the figure

$$\frac{\mathrm{PE}}{\mathrm{P'E}} = \frac{\mathrm{O'M}}{r'} = \frac{\mathrm{OM}}{r}$$

or

$$\mathrm{OM} = -r\frac{dy}{dr} = \mathrm{R}.$$

Hence, at any point P the tangent to the wave column passes through the lower extremity of the diameter of the rolling circle of centre P.

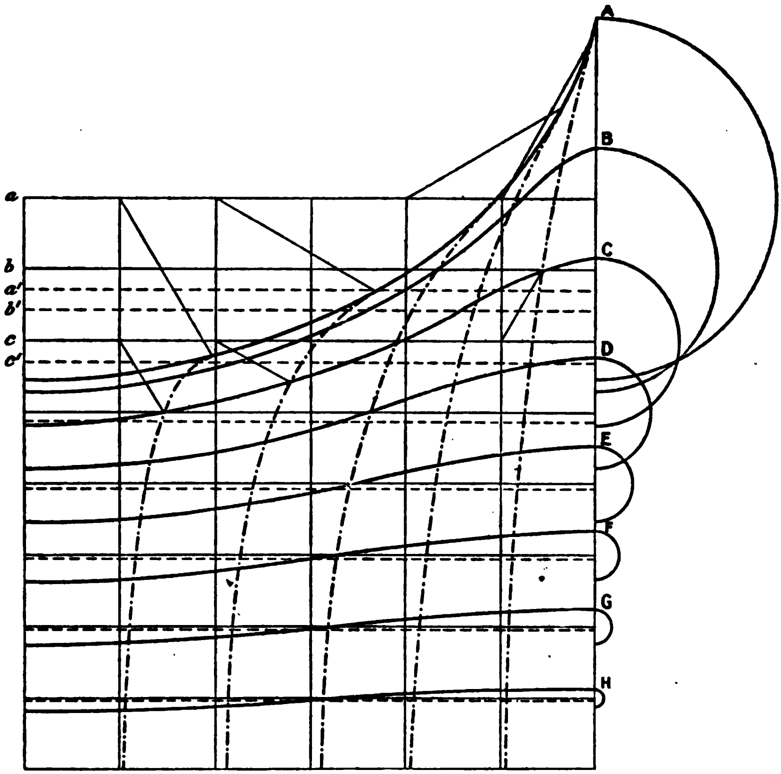

FIG. 34.

In addition to the wave columns, the sub-surface streams are lines of constant pressure. If the whole mass of the water, when still, be conceived to be divided into horizontal layers and into

vertical columns, the serpentine rolling of each layer, when agitated by waves, may be compared to that of a sheet laid upon the ground and shaken up and down at one edge. The motion of the columns may be compared to the bending and swaying of the stalks in a wind-swept field of corn; with the addition that each column of the fluid becomes alternately taller and slenderer, and shorter and thicker; being taller and slenderer while it is bending forwards, and shorter and thicker when swaying backwards.

The method of drawing these wave columns and surfaces of constant pressure is shown in Fig. 34. A number of orbit centres are taken at regular increasing depths, and the successive radii are obtained either graphically (Fig. 35) or by calculation. Each trochoid is drawn in the manner shown, and so the wave columns are immediately plotted. The tangent at any point of these wave columns passes through the lowest point of the rolling circle corresponding to that point. In a cycloidal wave, in which $r = R$, the wave columns are tangential to the cycloid.

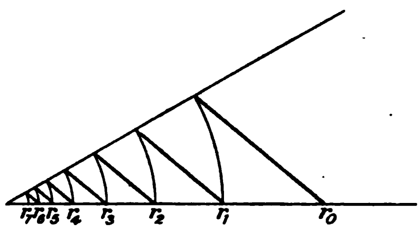

Fig. 35.

§ 32. Potential and Kinetic Energy in a Trochoidal Wave System.—In a trochoidal wave system, the potential energy is the amount of energy in virtue of the general elevation of the wave system being raised above the level in still water; and the kinetic energy is the energy stored up in all the elements of the water rotating about different centres.

The potential energy may be calculated as follows:—Referring to Fig. 33, let ds be the length of a small arc at P; taking moments of the different elements of water in the tube PQ, above the line of centres, the height of the centre of gravity of the water in the stream tube is

$$\frac{\int tds\,.\,r\cos\theta}{\int tds} = \frac{ntr\int^{\pi}\cos\theta d\theta}{nt\int_{0}^{\pi} d\theta} = 0.$$

This equation shows that when the wave system subsides, it falls to the level of still water.

In Fig. 34, the horizontal dotted lines a', b', c' . . . are the lines of orbit centres at equal distances apart. The lines A, B, C . . . represent the trochoidal stream lines surfaces corresponding to the line of centres a, b, c . . . A trochoidal wave is sharper at the crest than at the hollow; consequently, the still water level must be below the corresponding line of centres.

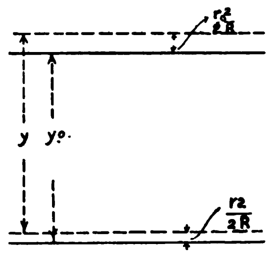

FIG. 36.

The difference may be estimated from equations (1) and (7), § 30, namely—

$$rdr + Rdy = \text{constant} = nt$$

and § 29,

$$nt = Rdy_o$$

$$\therefore rdr + Rdy = Rdy_o$$

$$\frac{r^2 - r_o^2}{2} + R(y - o) = R(y_o - o)$$

or

$$y - y_o = -\frac{r^2}{2R} + \frac{r_o^2}{2R}.$$

At an infinite depth, $r = 0$, and $y - y_o$ is the height of the orbit centres above still-water level; and is, therefore $\frac{r_o^2}{2R}$ (Fig. 36). Hence, the distance of all the other orbit centres are at a distance of $\frac{r^2}{2R}$ below the corresponding line of centres. The still-water levels corresponding to A, B, C . . . are indicated by a', b', c' . . .

Thus when the wave subsides, the centre of gravity of the water in the stream falls through the distance $\frac{r^2}{2R}$. Therefore, if b be the breadth of system,

$$\text{Potential energy} = bw\int ldy_o \cdot \frac{r^2}{2R}$$

$$= b\frac{w\pi}{R}\int (rdr + Rdy)r^2$$

$$= b\frac{w\pi}{R}\int_{r_o}^{o} r^3 dr + b\frac{w\pi}{R}\int_{o}^{\infty} r_o^2 R\epsilon^{-\frac{2y}{R}} dy$$

$$= b\frac{w\pi}{R}\left(-\frac{r_o^4}{4} + \frac{R^2 r_o^2}{2}\right)$$

$$= b\frac{w\pi R r_o^2}{2} - b\frac{w\pi r_o^4}{4R}$$

$$= b\frac{wlh^2}{16} - b\frac{w\pi^2 h^4}{32l}.$$

Again, the kinetic energy is the energy of all the particles of water moving in circular orbits around their respective centres, the whole mass of fluid being considered. The simplest way of finding the kinetic energy is to consider an element (Fig. 37) at P of weight

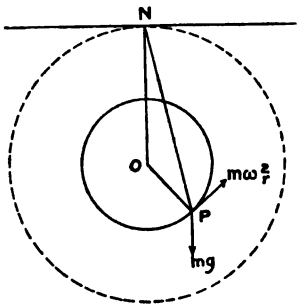

FIG. 37.

dw in the progressive wave. For the element the kinetic energy is

$$dw \cdot \frac{\omega^2 r^2}{2g}$$

and, therefore, for the whole tube it is

$$W\frac{\omega^2 r^2}{2} = W\frac{r^2}{2R}$$

where W is the weight of water in the tube. This is equal to the potential energy.

Thus in a trochoidal wave system the total energy is given by the expression

$$\frac{bwh^2}{8}\left(l - \frac{\pi^2 h^2}{2l}\right).$$

For a layer of thickness t, breadth b, the potential energy is

$$wbtl \cdot \frac{r^2}{2R}$$
$$= \pi wbtr^2$$

= weight of water in the cylinder of radius r multiplied by the thickness. This is independent of length of wave.

Conclusions.—The only assumption made in the course of the investigation is that the wave form is a trochoidal curve. But it has been demonstrated that it is a perfect dynamical problem. It will be found in Rankine's "Collected Papers."

The question of whether the motion is irrotational or rotational is a matter of small importance. As a matter of fact, it cannot, from a hydrodynamical point of view, be generated from rest. The true irrotational wave of the same height would fall within the trochoidal wave, being sharper at the crest and flatter at the trough; but the difference depends on the small quantity $\frac{r^3}{R^3}$. The velocity is slightly greater than that of a trochoidal wave (see § 38). But considering the many disturbing causes which ocean waves are subjected to, the results are sufficiently accurate, and have been verified by observing actual sea waves.

§ 33. **Observations on Waves.**—In observing waves three things have to be measured, namely, (1) the velocity, (2) the length, (3) the height.

The methods generally adopted in measuring the velocity and length is to measure the interval of time between which two suc-

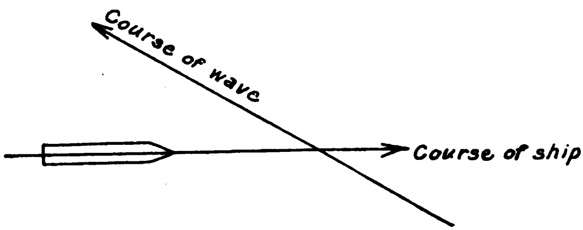

Fig. 38.

cessive wave crests reached, say, the stern, and also the time that the crest of the first wave took to run the length of the ship.

The length of the ship, her speed and course relative to the direction of the waves, the velocity and length of the wave, can be readily calculated as follows :—Assume the sea to be a head one on the ship (Fig. 38) and inclined at an angle a with the direction of the wave's progress.

Let l = length of the ship in feet;
s = speed of the ship in feet per second;
t = the time taken for a wave crest to run the length of the boat;
t' = time between which successive waves cross the stern.

Then, the velocity of wave $= \dfrac{l \cos a - st \cos a}{t}$

$$= \left(\frac{l}{t} - s\right) \cos a$$

$$= w, \text{ say.}$$

The length of the wave is distance run by wave + projected distance run by ship.

$$= wt' + st' \cos a$$

$$= t' \cos a \left(\frac{l}{t} + s - s\right)$$

$$= l \cos a \frac{t'}{t}.$$

For a following sea, the sign of s must be reversed.

Thus, from the observed data, the length and velocity can be obtained. The results can then be compared with the trochoidal theory.

The determination of the third element, namely, the height, is a much more difficult matter. One method is to take up a position such that, when the ship is in the trough and for an instant upright, the successive ridges as viewed by the observer first reach the line of the horizon without observing it. The height of the eye above the water level correctly measures the height of the wave. Such a method is liable to give rise to considerable inaccuracy. In making such observations, it is advisable to select a position as near as possible amidships, so that the influence of

pitching and ascending is reduced. Allowance must be made for rolling. If the boat is not exactly in the trough, considerable error creeps in. Another method is to use a delicate aneroid. The difficulty arose in estimating the height of the eye above sea-level at the moment of observation. When the ship was at its lowest point, the surface of the water might be 10 feet below the eye; but when the crest of the wave rushed past it might be only 1 foot. To the aneroid reading must be added 9 feet. The largest wave[1] observed was in the North Atlantic, and had a length of half a mile from crest to crest; its period was 53 seconds. The largest observed in European waters are said to have a period of $19\frac{3}{4}$ seconds, corresponding to a theoretical length of some 2000 feet.

Sir William White (to whom reference must be made) gives the following table, showing the ratio of height to length:—

	Length ÷ height.		
	Maximum.	Average.	Minimum.
100 feet and under	30	17	5
100–200	40	20	9
200–300	40	25	10
300–400	40	27	17
400–500	40	24	15
500–600	40	23	17

Waves in Shallow Water

In shallow water of uniform depth, the orbit of each particle becomes an oval of a height less than its length, and nearly, but not exactly, of an elliptical section. The oval orbits are smaller and more flattened the nearer the particles are to the bottom; and the particles in contact with the bottom oscillate in straight lines, unlike those in trochoidal waves, which have no motion. In dealing with waves in shallow water, the elliptical orbit is always assumed.

[1] Sir William White, K.C.B., F.R.S., "Manual of Naval Architecture."

§ 34. **Wave-forms.**—In Fig. 39, the two dotted circles *a*, *b* represent the major and minor auxiliary circles. The circle *c* is what may be termed the rolling circle, rolling on the line *d*. Draw a radius OA at any angle, and meeting the circles *a* and *b* in A and B respectively. Through A and B draw vertical and horizontal lines meeting in P. Then P is a point in the wave; and if the ellipse be drawn, it represents the orbit path of the particle P. Now imagine the circle *c* to roll on the line *d* so that the arm OB rotates about O; and let the major axis of the ellipse be always horizontal. Also let the circle *c* roll along the line *d* so that any radius OB has a motion of rotation about O, and a displacement along a direction parallel to the line *d*—the length of the wave being the circumference of the circle *c*. The locus of B will be different from a trochoid. If a trochoid be plotted of the same length and height (being equal to the minor axis) then the "elliptic trochoid" (as it may be termed) will be sharper at the crest and fuller at the hollow than the true trochoid; and will be entirely below it.

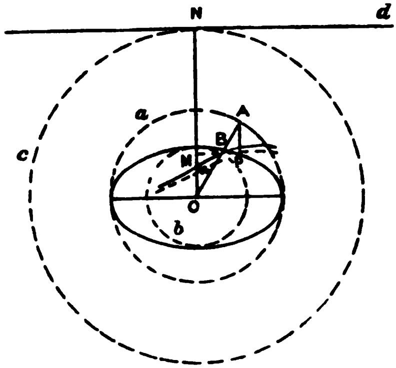

FIG. 39.

§ 35. **Velocity of Propagation.**—Let a and b be the semi-major and semi-minor axes, OAB a radius of the auxiliary circles making an angle θ with the vertical radius ON of the rolling circle (Fig. 39). The point P on the wave surface is obtained by drawing lines AP, BP vertically and horizontally to meet in P. Thus, taking the crest as origin—the horizontal line through O as axis of x and the vertical line ON as axis of y—and the radius of the rolling circle ON being R, then

$$\text{OM} = b \cos \theta, \quad \text{PM} = a \sin \theta$$

and, referred to the crest C as origin, the co-ordinates of P are

$$x = \text{R}\theta - a \sin \theta, \quad y = b - b \cos \theta.$$

If ω be the angular velocity of the rolling circle about its centre, then

$$\dot{x} = (\mathrm{R} - a\cos\theta)\omega$$

$$\dot{y} = b\sin\theta\,.\,\omega.$$

The velocity of P, therefore, is

$$v^2 = (\mathrm{R}^2 - 2a\mathrm{R}\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta)\omega^2.$$

Hence, if suffix o refer to the crest,

$\dfrac{p - p_o}{\sigma} = \dfrac{v_o^2 - v^2}{2g} + z_o - z,$ $z_o\,.\,z$ being the elevations of the crest and the point P above the lines of centre

$$= \frac{\omega^2}{2g}\{(\mathrm{R} - a)^2_{\theta=o} - (\mathrm{R}^2 - 2a\mathrm{R}\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta)\}$$
$$+ b(1 - \cos\theta)$$

$$= \frac{\omega^2}{2g}\{(1 - \cos\theta)(b \times \frac{2g}{\omega^2} - 2a\mathrm{R}) + \sin^2\theta\,(a^2 - b^2)\}.$$

Thus on the assumption made, the assumption of uniform surface pressure cannot be fulfilled; but if

$$\omega^2 = \frac{b}{a}\cdot\frac{g}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

the pressures at the crest and trough—since the second is zero—will be the same; in which case

$$\frac{p - p_o}{\omega} = \frac{\omega^2}{2g}(a^2 - b^2)\sin^2\theta$$

On the assumption of equality at the crest and trough, the result may be obtained more shortly as follows.

At the crest, velocity $= (\mathrm{R} - a)\omega$

trough $= (\mathrm{R} + a)\omega$

$$\therefore\ (\mathrm{R} + a)^2\omega^2 - (\mathrm{R} - a)^2\omega^2 = 2g\,.\,2b$$

$$\omega^2 = \frac{b}{a}\cdot\frac{g}{\mathrm{R}}.$$

The velocity of propagation is

$$V = \omega R = \sqrt{\frac{b}{a} g R} = \sqrt{\frac{glb}{2\pi a}}$$

where l is the length of the wave in feet.

The velocity of a wave in shallow water is thus less than the velocity in deep water of the same length.

§ 36. **Variation of Orbit Radii with Depth.**—The next step is to find how the dimensions of the orbit centres varies with the depth. Since the relations obtained in the preceding article are only true for the crest and trough sections, it is only necessary to consider these sections. Referring to Fig. 40, the full lines represent, say, the surface conditions; the dotted lines represent those of a sub-surface near to the surface line; the distance between the centres, OO′, being equal to the distance NN′, and the radii of the rolling circles being the same. The orbit axes are, however, different, the full line referring to the surface line, and the dotted line to the sub-surface one. Let dy, as before, be the distance OO′. Then

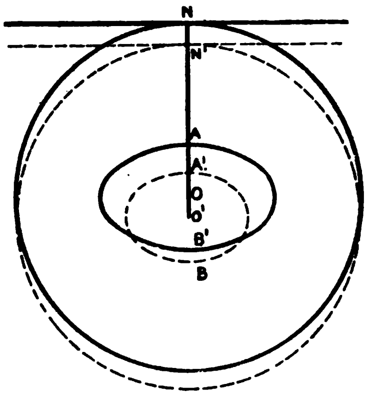

FIG. 40.

$$NN' = OO' = dy.$$

At the crest A, the thickness of the stream tube is

$$AA' = dy - db$$

and at the hollow

$$BB' = dy + db.$$

The velocity at the crest $= v_c = \omega(R - a)$

" trough $= v_t = \omega(R + a)$.

Also, from the equation of continuity

$$(dy - db)(R - a) = (dy + db)(R + a)$$

or
$$\frac{dy - db}{dy + db} = \frac{\mathrm{R} + a}{\mathrm{R} - a}$$

and, therefore,
$$\frac{dy}{db} = -\frac{\mathrm{R}}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

Again not only is the flow along a stream tube constant, but so also is the molecular rotation (§ 3), so that—

$$\frac{v_c}{\rho_c} + \frac{dv_c}{\mathrm{AA}'} = \frac{v_t}{\rho_t} + \frac{dv_t}{\mathrm{BB}'}.$$

Now
$$v_c = (\mathrm{R} - a)\omega\,; \quad v_t = (\mathrm{R} + a)\omega$$
$$dv_c = +\,\omega da\,; \qquad v_t = -\,\omega da$$
$$\mathrm{AA}' = dy - db\,; \quad \mathrm{BB}' = dy + db$$
$$\frac{1}{\rho_c} = \frac{b}{(\mathrm{R} - a)^2};^1 \quad \frac{1}{\rho_t} = -\frac{b}{(\mathrm{R} + a)^2}{}^1$$

whence
$$\frac{b\omega}{\mathrm{R} - a} + \frac{\omega da}{dy - db} = -\frac{b\omega}{\mathrm{R} + a} - \frac{\omega da}{dy + db}.$$

or
$$\frac{\dfrac{b}{\mathrm{R}}}{1 - \left(\dfrac{a}{\mathrm{R}}\right)^2} = \frac{-\dfrac{da}{dy}}{1 - \left(\dfrac{db}{dy}\right)^2} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Combining with (10),
$$\frac{da}{dy} = -\frac{b}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$
$$\frac{db}{dy} = -\frac{a}{\mathrm{R}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$
$$\frac{da}{db} = +\frac{b}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

If $b = a = r$, the equations (12) and (13) become

$$\frac{dr}{dy} = -\frac{r}{\mathrm{R}}$$ as in the trochoidal wave (§ 30).

§ 37. Determination of Axes.—Equation (14) gives

$$a^2 - b^2 = \text{constant} = a_o^{\,2} - b_o^{\,2}, \text{ at the surface.}$$

[1] Consult Professor Lamb's "Hydromechanics," pp. 409–414.

Equation (13) gives

$$\frac{d^2b}{dy^2} = -\frac{1}{R}\cdot\frac{da}{dy} = +\frac{b}{R^2}, \text{ from equation (12),}$$

whence $$b = A\varepsilon^{-\frac{y}{R}} + B\varepsilon^{+\frac{y}{R}}$$

To get the crests—h being the depth of the bottom below the line of orbit centres of the surface particles—when

$$y = 0, \qquad b = b_o$$
$$y = h, \qquad b = 0$$

the particles at the bottom moving to and fro in a straight line.

Hence

$$b = b_o\frac{\varepsilon^{\frac{h-y}{R}} - \varepsilon^{-\frac{h-y}{R}}}{\varepsilon^{\frac{h}{R}} - \varepsilon^{-\frac{h}{R}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

the depths being measured below the line of centres of the surface orbits. Again, in equation (14)—

$$a = -R\frac{db}{dy} = +b_o\frac{\varepsilon^{\frac{h-y}{R}} + \varepsilon^{-\frac{h-y}{R}}}{\varepsilon^{\frac{h}{R}} - \varepsilon^{-\frac{h}{R}}} \quad . \quad . \quad . \quad (16)$$

whence $$\frac{b}{a} = \frac{\varepsilon^{\frac{h-y}{R}} - \varepsilon^{-\frac{h-y}{R}}}{\varepsilon^{\frac{h-y}{R}} + \varepsilon^{-\frac{h-y}{R}}}$$

$$= \tanh\left(\frac{h-y}{R}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This equation enables the ratio of axes at any depth y below the line of centres to be calculated.

At the surface

$$\frac{b_o}{a_o} = \frac{\varepsilon^{\frac{h}{R}} - \varepsilon^{-\frac{h}{R}}}{\varepsilon^{\frac{h}{R}} + \varepsilon^{-\frac{h}{R}}} = \tanh\frac{2\pi h}{l} \quad . \quad . \quad . \quad . \quad (18)$$

Hence, knowing the depth of water (below the line of centres)

and the length of the wave, the ratio of $\frac{b_o}{a_o}$ can be obtained from equation (18), and the ratio for any other depth from equation (17).

Thus the equation—

$$V^2 = \frac{b_o}{a_o} g R$$

gives the velocity of propagation in a wave of given length and depth of water.

The horizontal oscillation at the bottom is

$$2\sqrt{a_o^2 - b_o^2} = \frac{4b_o}{\varepsilon^{\frac{h}{R}} - \varepsilon^{-\frac{h}{R}}} \quad . \quad . \quad . \quad . \quad . \quad (19)$$

At the surface—

$$\frac{b_o}{a_o} = \frac{\varepsilon^{\frac{4\pi h}{l}} - 1}{\varepsilon^{\frac{4\pi h}{l}} + 1} = \frac{\frac{4\pi h}{l} + \frac{8\pi^2 h^2}{l^2}}{2 + \frac{4\pi h}{l}} \text{ approximately}$$

$$= \frac{2\pi h}{l}$$

$$= \frac{h}{R}$$

if $\frac{h}{l}$ be small.

Whence

$$V^2 = \frac{b_o}{a_o} . gR = gh \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where h is the depth of water.

Again, making the same assumption, equation (16) gives

$$a = b_o \frac{1}{\frac{h}{R}} = b_o \frac{R}{h} = \text{constant for all depths}$$

and from equation (15)—

$$b = b_o . \frac{h - y}{h}$$

so that vertical displacement is proportional to the distance from

the bottom. This represents the case of the wave of translation (§ 25).

Again, if h be very large, equation (17) shows that $b = a$, since the second term in the numerator and denominator is very small. All the orbits are thus circular. From equation (15)—

$$b = b_o \varepsilon^{-\frac{y}{R}}$$

and from equation (8)—

$$\omega^2 = \frac{g}{R}.$$

These represent the *oscillating waves in deep water* (§ 26).

Thus the oscillating wave in shallow water is the intermediary system between oscillating waves in deep water and the solitary wave of translation. The three types may be illustrated by observing large waves out at sea coming toward the shore. As the waves approach the shore, the front of each wave becomes steeper and steeper than the back, and at length curls forward and falls over. After a wave has broken on the shore, it does not cease to travel, but if the slope be gentle, the beach shallow and very extended, the whole inner portion of the beach is covered with positive waves. This accounts for the phenomena of breakers transporting shingle and other substances shorewards after a certain point. At a great distance from the shore, or where the shores are deep and abrupt, the wave is of the second order (even in shallow water), and a body floating near the surface is alternately carried forward and backward by the waves, neither is the water affected to a greater depth; whereas near the shore the whole action of the wave is inwards, and the force extends to the bottom of the water and throws shingle shorewards; hence the abruptness also of the shingle and sand near the margin of the shore where the breakers usually run.

If the waves breaking on a long, shelving beach be observed, it will be noticed that the shore is oblique to the direction of waves at sea; they invariably come in parallel to the shore. The reason for this is that at the end of the shore farthest out at sea the water first comes on the shallow part, and—being a

a wave of translation—has its velocity reduced; whilst the further end, in deep water, has a great velocity of propagation. Thus the waves wheel round the sea end of the shore.

It is of interest to determine the depth where the trochoidal ceases to be approximate. If, for example, the error in velocity for a given length is one per cent., then the ratio of depth to length is about one-half. Thus, for all depths greater than one-half length, the trochoidal theory many be taken.

§ 38. **Molecular Rotation.**—As in a trochoid, so in the waves just discussed—the system cannot be generated from rest. The value of the molecular rotation may be calculated at the crest, and is (§ 3)

$$\frac{1}{2}\left\{\frac{b\omega}{\mathrm{R}-a}+\frac{\omega da}{dy-db}\right\}$$

$$=\frac{1}{2}\left\{\frac{b\omega}{\mathrm{R}-a}+\frac{\omega}{\frac{dy}{da}-\frac{db}{da}}\right\}$$

$$=\frac{1}{2}\left\{\frac{b\omega}{\mathrm{R}-a}+\frac{\omega}{-\frac{\mathrm{R}}{b}-\frac{a}{b}}\right\}$$

$$=\frac{1}{2}\frac{\omega ah}{\mathrm{R}^2-a^2}.$$

$$\text{For a trochoidal wave } a=r=\frac{h}{2};\text{ and}$$

$$\therefore\text{ molecular rotation }=\frac{\omega r^2}{\mathrm{R}^2-r^2}.$$

§ 39. **Group Velocity.**[1]—It has often been noticed that when an isolated group of waves is advancing over relatively deep water the velocity of the group as a whole is less than that of the waves comprising it. If attention be fixed on a particular wave, it is seen to advance through the group, gradually dying out as it approaches the front, whilst its former position in the group is occupied in succession by other waves which have come forward

[1] See a paper by Professor Osborne Reynolds, F.R.S., *Nature*, August 23, 1877, read before Section G of the British Association.

from the rear. Or again, when a stone is thrown on the surface of a pond, the series of rings which it causes gradually expand so as to finally embrace the entire surface of the water; but if careful notice be taken it is seen that the waves travel outwards at a considerably greater rate than that at which the disturbance spreads.

But perhaps the most striking manifestation of the phenomenon is in the waves which spring from the bows of a rapid boat, and attend it on its course. A wave from either bow extends backwards in a slanting direction for some distance and then disappears; but immediately behind it has come into existence another wave, parallel to the first, beyond which it extends for some distance, when it also dies out, but not before it is followed by a third, which extends still farther, and so on, each wave overlapping the others rather more than its predecessor. Although not obvious, very little consideration serves to show that the stepped form of these columns of waves is a result of the continual dying out of the waves in front of the group, and the formation of fresh waves behind. For, as each wave cuts slantwise through the column formed by the group, one end is on the advancing side or front of the group, and this is continually dying, while the other is in the rear end, and is always growing.

§ 40. **Dynamical Illustration.**—The propagation of a wave form through a medium need not necessarily involve the transmission of energy. This kind of wave may be well understood by suspending a series of small balls by threads, so that the balls all hang in a row, and the threads are all of the same length. If the finger be run along, so as to set the balls oscillating in succession, the motion will be such as to give the idea of a series of waves propagated from one end to the other; but in reality there is no propagation—each pendulum swings independently of its neighbours. There is no communication of energy, the wave being merely the result of the general arrangement of the motion.

In this case there is no communication of energy, neither is there any propagation of disturbance. Any one ball may be set swinging without in the least disturbing the others; and what is indicated here is a general law, that wherever a disturbance is transmitted through a medium by waves, and therefore a

propagation of disturbance, there must always be communication of energy. The rate at which energy is transmitted in different media, or by different systems of waves, is very different. This may be illustrated by experiment. If the balls just described are all connected by an elastic thread (Fig. 41), they can no longer swing independently. If one be set in motion,

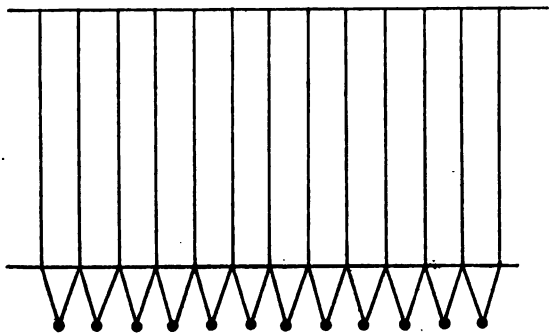

Fig. 41.

then, on account of the connecting thread, it will communicate motion to its neighbours until they swing with it, so that now waves would be propagated through the balls. The rate at which a ball would impart motion to its neighbours would clearly depend on the tension of the connecting thread. If this was very slight compared with the weight of the balls, it would stretch, and the ball might accomplish several swings before it had set its neighbours in full motion, so that of the initial energy of disturbance a very small portion is transmitted at each swing. But if the tension of the thread be great compared with the weight of the balls, one ball cannot be disturbed without causing a similar disturbance in its neighbours, and then the whole energy will be transmitted. This is illustrated by laying a rope or chain on the ground, and fastening down one end; if the loose end be caused to oscillate in a vertical plane, the oscillations will be transmitted to the fixed end, and the chain or rope will lie quiescent on the ground. Thus in this case all the energy is transmitted.

The straight cord and the pendulums represent media in which the waves are at the opposite limits—in one case, none of the

energy of disturbance is transmitted, and, in the other, the whole is transmitted. Between those two limits there may be waves of infinite variety, in which any degree of energy, from all to nothing, is transmitted. In waves of sound all the energy is transmitted; in waves of water it is analogous to the waves in the balls suspended when connected by an elastic string.

§ 41. **Calculation of Group Velocity.**—In regular trochoidal waves the particles move in vertical circles with a constant velocity, and are always subjected to the same pressure. Of the energy of disturbance, half goes to give motion to the particles and half to raise them from the initial position to the mean height which they occupy during the passage of the wave (§ 32).

The mean horizontal positions of the particles remain unaltered by the waves, hence, since their velocities are constant, none of their energy of motion is transmitted; nor, since the pressure on each particle is constant, can any energy be transmitted by pressure. The only energy, therefore, which remains to be transmitted is the energy due to elevation, and that this is transmitted is obvious, since the particles are moving forward when above their mean position, and backward when below it. This energy constitutes half the energy of disturbance; and this is, therefore, the amount transmitted.

A mathematical proof of the statement: *In waves in deep water the rate at which energy is carried forward is half the energy of disturbance per unit length multiplied by the rate of propagation.*[1] Let h_0 be the initial height occupied by a particle supposed to be of unit weight, h_1 the height of the centre of the circle in which it moves as the wave passes, r the radius of the orbit, and θ the angle which the radius vector makes with the horizontal diameter; then the height of the particle above its initial position is $h_1 - h_0 + r \sin \theta$. Adding to the height due to velocity, the whole energy of disturbance is

$$2(h_1 - h_0) + r \sin \theta.$$

The velocity of the particle is

$$\sqrt{2g(h_1 - h_0)} \quad (\S\ 32)$$

[1] Professor Osborne Reynolds, F.R.S., *Nature*, August 23, 1877, pp. 343, 344.

and the horizontal component of this is

$$\sqrt{2g(h_1 - h_0)} \sin \theta.$$

Therefore the rate at which energy is being transmitted by the particle is

$$\{2(h_1 - h_0) + r \sin \theta\}\sqrt{2g(h_1 - h_0)} \sin \theta$$

and the mean of this is

$$\frac{1}{2\pi}\int_o^{2\pi} \{2(h_1 - h_0) + r \sin \theta\}\sqrt{2g(h_1 - h_0)} \sin \theta d\theta$$

$$= \tfrac{1}{2} r\sqrt{2g(h_1 - h_0)}$$

and if l be the length of the wave, and nl the rate of propagation,

$$h_1 - h_0 = \frac{\pi r^2}{l} \quad \text{and} \quad \frac{2g}{l} = 4\pi n^2.$$

Therefore the mean rate at which energy is transmitted by the particle is

$$nl(h_1 - h_0)$$

or the rate of propagation is half the energy of disturbance.

It now remains to consider the speed of the groups of waves, and to prove that *if the rate at which energy is transmitted is equal to the rate of propagation multiplied by half the energy of the disturbance, then the velocity of the group is half that of the individual waves.*

Let P_1, P_2, P_3, P_4 be points similarly situated in a series of waves which gradually diminish in size and energy of disturbance from P_3 to P_1, in which direction they are moving. Let E be the energy of disturbance between P_1 and P_2 at time t; $E + a$ the energy between P_2 and P_3, $E + 2a$ between P_3 and P_4, and so on. Then at time $t + \frac{1}{n}$ the wave has moved through one wave-length; it follows that the energy between P_1 and P_2 will be

$$\frac{E + E + a}{2} = E + \frac{a}{2}$$

and between P_2 and P_3 will be

$$\frac{E + a + E + 2a}{2} = E + \frac{3a}{2}.$$

And again, after an interval $\frac{1}{n}$, the energies between P_1, P_2, P_3 will be respectively

$$\frac{E+\frac{a}{2}+E+\frac{3a}{2}}{2}=E+a$$

and
$$\frac{E+\frac{3a}{2}+E+\frac{5a}{2}}{2}=E+2a.$$

So that, after the waves have advanced through two wave-lengths, the distribution of the energy will have advanced one, or the speed of the group is half that of the individual waves.

The above investigation refers to deep-water waves. As the waves enter shallow water the motion of the particles becomes elliptical, the eccentricity depending on the depth of water; and it may be shown that under these circumstances the rate at which energy is transmitted is increased until, when the elliptic paths approach to straight lines, the whole energy is transmitted; and, consequently, it follows that the rates of the speed of the groups to the speed of the waves will increase as the water becomes shallower, until they are sensibly the same. In a wave of translation all the energy is transmitted.

§ **42. Combination of Wave Systems.**—A group of waves in deep water may be imagined to consist of two component systems, but slightly different in length (Fig. 42).

Let l_1, l_2 be the two wave-lengths. If the crests coalesce at A, hen the n^{th} crest to the right or left of A of the longer system is

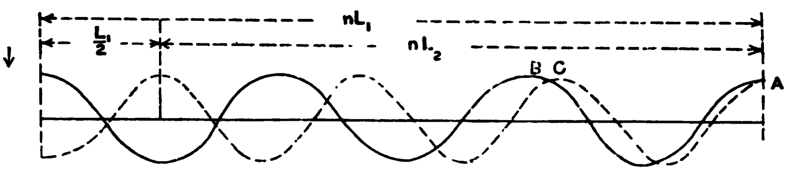

Fig. 42.

distant $n(l_1 - l_2)$ from the n^{th} crest of the shorter system. When the crest of the shorter system coincides with the hollow of the longer system, the distance moved through is $\frac{l_1}{2}$ (Fig. 42); and,

therefore, from the figure

$$n = \frac{l_1}{2(l_1 - l_2)}$$

and the semi-group length $= \dfrac{l_1 l_2}{2(l_1 - l_2)}$ . . . . (21)

Moreover, if v_1 and v_2 be the velocity of the two systems, and at any instant the crests are coincident at A, clearly B will coincide with C after

$$\frac{l_1 - l_2}{v_1 - v_2} \text{ seconds.}$$

In which case, the distance moved through by the centre of the system will be

$$v_1 t - l_1$$

and the velocity of the system will be

$$v_1 - \frac{l_1}{t} = v_1 - \frac{l_1(v_1 - v_2)}{l_1 - l_2} = \frac{v_2 l_1 - v_1 l_2}{l_1 - l_2}.$$

In a trochoidal system

$$l_2 = \frac{v_2^2}{g \,.\, 2\pi} \quad \text{and } l_1 = \frac{v_1^2}{g \,.\, 2\pi} \quad (\S\ 28)$$

so that the velocity of the centre of the system

$$= \frac{v_1 v_2}{v_1 + v_2} \quad . \ . \ . \ . \ . \ . \ . \ . \quad (22)$$

Now, to realize the conception of a "group" of waves, the lengths of the waves of the two systems must be imagined practically equal to each other. In that case the length of the group is infinite (equation (21)) and the velocity of the group is $\frac{v}{2}$, where v is the velocity of the wave of each system.

Thus the group has the highest crest at the centre, and the height of the waves diminishes gradually and becomes zero at an infinite distance in either direction. The centre of that group moves forward at half the speed of the individual waves.

§ 43. Combination of Waves of Equal Lengths but Different Amplitudes.—Consider two waves, moving in the same direction, of the same length, and, therefore, of the same velocity, but let

them be of different heights. Let λ be the wave-length, h_1, h_2 the heights of the waves, and s be the phase difference. The motion will be assumed trochoidal.

Let (Fig. 43) O be the orbit centre of the surface particles of

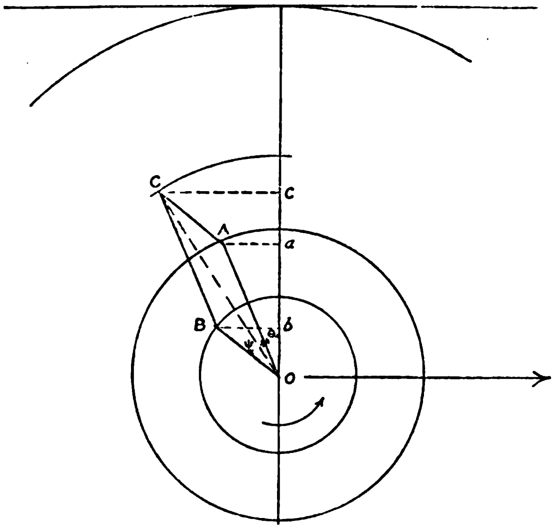

FIG. 43.

either component. The orbit centres at the surface are not on the same level, but the difference is slight. Let AOB be the constant difference of phase angle, so that A describes the one trochoid and B the other. At the instant considered, the height of the particle A above O is Oa, of the particle immediately below, in the second wave, is Ob. The resultant height is Ob + Oa, which may be found by drawing the parallelogram OACB, and projecting OC on the vertical line through the centre O. If the parallelogram OACB revolves uniformly about O, whilst O moves uniformly along the horizontal line—the two speeds being equal and depending on the length of the wave—A will trace out the profile of the first wave, B of the second, and C of the third. The semi-height of the resultant system is OC, so that, if h be the height of the resultant system,

$$h^2 = h_1^2 + h_2^2 + 2h_1h_2 \cos \psi$$

ψ being the angle AOB.

Fig. 44 shows the two component systems A and B, and the resultant system C, at their proper phase difference. The position defined by Fig. 43 corresponds to the section *cabo* in Fig. 44.

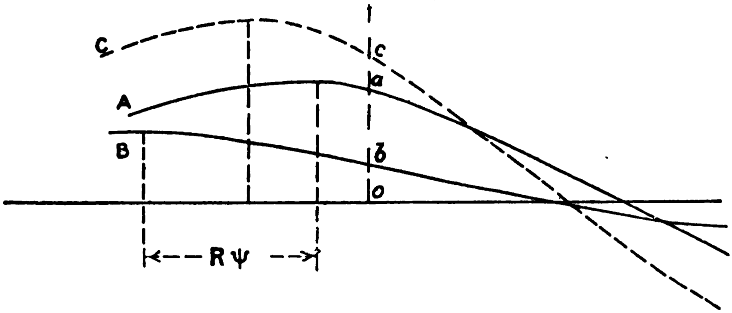

FIG. 44.

If s' be the distance between any two adjacent crests

$$s' = R\psi = \frac{\lambda\psi}{2\pi} \qquad \therefore \psi = \frac{2\pi s'}{\lambda}$$

$$\therefore h^2 = h_1{}^2 + h_2{}^2 + 2h_1h_2 \cos \frac{2\pi s'}{\lambda}.$$

§ 44. **Ripples, or Capillary Waves.**—In the waves so far considered it has been assumed that gravity is the sole origin of the motive forces. This is practically so in the case of sea waves, and in the case of waves formed by a body moving through the water at a high speed. But in the case of waves created by the wind, or by a body—such as a fishing-rod—drawn through the water at a slow speed, gravity is not the sole origin of the motive forces, but cohesion or surface tension bears a considerable part.

The effect of cohesion on waves is seen by observing a body, such as a canoe, moving through the water at a slow speed. A set of very short waves advancing will be noticed in front of the body, and another set of waves, considerably larger, advancing in its wake. The two sets of waves advance each at the same rate as the moving body; so that there are two different wave-lengths which give the same velocity of propagation. When the speed of the body is increased, the waves preceding it become shorter and those in the wake become longer.

The stern waves move with exactly the speed of the vessel and appear to be of such length as to verify the ordinary formula for deep-sea waves. In these waves gravity is assumed as the sole origin of the motive forces. The following investigation will show the part taken by ripples.

The effect of cohesion on surface tension is to put the surface layer in tension. Thus, the pressure is greater at the crest and less at the hollow than it otherwise would be. The tension is reckoned in pounds per foot length, and will be denoted by T. Thus, if ρ be the radius of curvature at any point, the pressure due to surface tension is

$$p = \frac{T}{\rho}.$$

Now, assume the profile of the wave to be a trochoid of small height, so that (Fig. 45)

$$x = R\theta, \quad \text{approximately}$$

$$y = r - r\cos\theta = r\left(1 - \cos\frac{x}{R}\right).$$

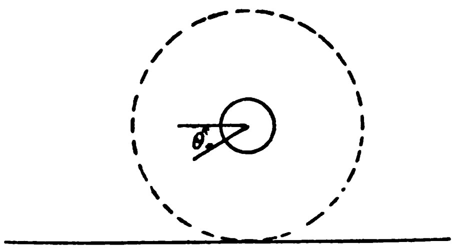

FIG. 45.

In that case

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} = \frac{r}{R^2}\cos\frac{x}{R} = \frac{r}{R^2}\cos\theta.$$

Also, assuming the wave trochoidal, let suffix c refer to the crest. Then for any other point

$$\frac{v^2}{2g} - \frac{v_c^2}{2g} + r\cos\theta - r = \frac{p_c - p}{w} \quad . \quad . \quad . \quad (23)$$

or, using the previous results for a trochoidal wave,

$$\frac{(R^2 + r^2 - 2Rr\cos\theta)}{2g}\omega^2 - \frac{(R - r)^2\omega^2}{2g} + r\cos\theta - r = \frac{p_c - p}{w}$$

or

$$r(1 - \cos\theta)\left\{\frac{\omega^2}{g}R - 1\right\} = \frac{p_c - p}{w} \quad . \quad . \quad . \quad (24)$$

Now, the difference of pressure is due only to surface tension, and this is equal to

$$T\left(\frac{1}{\rho_c} - \frac{1}{\rho}\right).$$

As the capillary waves are of small height, the equation may be written—neglecting r^2 and higher powers of r—

$$\frac{\omega^2 R}{g} - 1 = \frac{T}{wR^2}$$

$$\omega^2 = \frac{g}{R} + \frac{Tg}{wR^3}$$

or

$$V^2 = gR + \frac{Tg}{wR} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$= \frac{gl}{2\pi} + \frac{2\pi gT}{wl}$$

V being the velocity of the wave.

For a given velocity, this is a quadratic in R, giving two wave-lengths.

For distilled water at 60 F.,

$$T = 0{\cdot}074 \text{ grammes per centimetre}$$

$$= \frac{0{\cdot}074 \times 2{\cdot}205}{1000} \times \frac{12}{0{\cdot}3937} = 0{\cdot}00496 \text{ pounds per square foot,}$$

whence, for such a case, equation (25) becomes

$$V^2 = 5{\cdot}14l + \frac{0{\cdot}016}{l} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

The quadratic in l gives

$$l^2 - 0{\cdot}195V^2 l = -\ 0{\cdot}0031$$

whence

$$l = 0{\cdot}0975V^2 \pm \sqrt{0{\cdot}0095V^4 - 0{\cdot}0031}.$$

Thus l is imaginary provided

$$V^4 < 0{\cdot}327$$
$$V < 0{\cdot}76 \text{ feet per second}$$
$$< 9{\cdot}1'' \text{ or } 23 \text{ cm. per second}$$
$$< \tfrac{1}{2} \text{ mile an hour, about,}$$

in which case $l = 0{\cdot}055$ foot $= 0{\cdot}67$ inch.

The relative values between the gravity and cohesion terms are expressed in equation (26). The former increases with l, the latter decreases. When l corresponds to the limiting velocity, these two terms are equal, namely, 0·286 foot per second.

The waves whose length is less than that corresponding to the limiting velocity are called *ripples*. The maximum length of a ripple is 0·67 inch, which occurs when they are first formed, the body moving through the water at a speed of half a mile an hour (about). As the velocity is increased, l (taking the negative sign) decreases, becoming ultimately zero when V^4 is great compared to 0·327. This result is confirmed by the crowding of the front waves as the speed increases. The ripple form is approximately hyperbolic. If the speed is diminished towards the critical velocity, the ripples in front elongate and become less curved, and the waves in the rear become shorter, till at the critical velocity waves and ripples seem nearly equal, and with ridges in straight lines perpendicular to the line of motion. Below that velocity no waves or ripples are created.

The general expression for the length of the wave, including gravity and cohesion effect, is

$$l = \frac{\pi V^2}{g} \pm \frac{\pi V^2}{g}\sqrt{1 - \frac{4g^2T}{wV^4}}$$

giving a critical velocity of

$$\sqrt[4]{\frac{4g^2T^2}{w}}$$

and a maximum length of ripple of

$$\frac{2}{\pi}\sqrt[2]{\frac{T}{w}}.$$

CHAPTER III

RESISTANCE OF SHIPS:

EDDY, SKIN, AND WAVE-MAKING RESISTANCE

§ **45. Surface Resistance.**—It has been pointed out in Chapter I. that in the stream line motion of a perfect fluid past an obstacle placed between parallel plates, the pressure and velocity at every point are determinable, and that there is no net force tending to drag the body along. At the boundaries there is slipping.

In the motion of an exceedingly viscous fluid past an object placed between plates very near together, the pressure and velocity along and perpendicular to the stream lines could be determined. Thus the motion in this case was also determinate. At the boundary there is no slipping and the stream lines are identical in form with those obtained for a perfect fluid, the motion being plane.

These results represent two ideal cases which cannot be obtained in practice. In all ordinary cases the motion is not a stream line motion, but there is a general motion of the fluid combined with a number of rotating eddies moving along with the stream which, on account of their high rate of distortion become rapidly absorbed, the energy necessary for their creation—which becomes converted into heat and, therefore, becomes unavailable—causing a diminution of pressure into the mass of the fluid.

Thus, in the flow past an immersed surface, and more so in a body floating on the water, there is considerable resistance to the motion. That force is due to the formation of eddies which extend to a certain distance, more or less indeterminate, from the body, completely surrounding it. It is usually called *surface friction.*

§ 46. **Eddy Resistance.**—If the body be not of "fair" form, but have blunt corners, the water is not directed round the corners, except in ideal cases, or in experiments made under peculiar

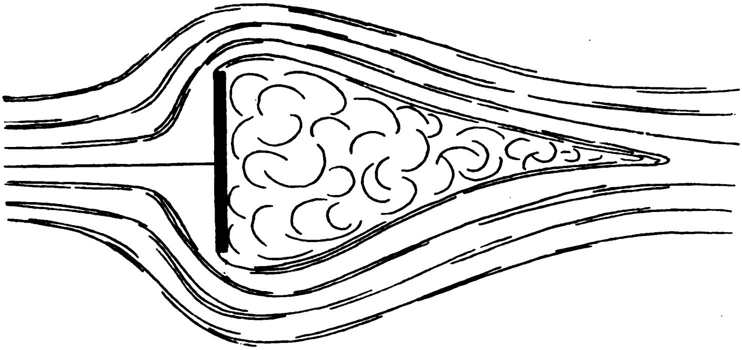

FIG. 46.

conditions, but the average motion of the water appears to be considerably modified. The water is abruptly thrown off from the solid body, and eddies (Fig. 46) are formed behind the plate, so

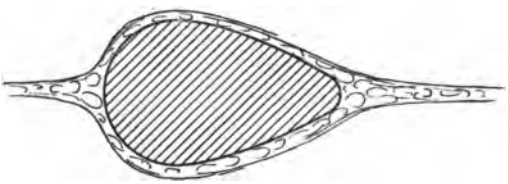

FIG. 47.

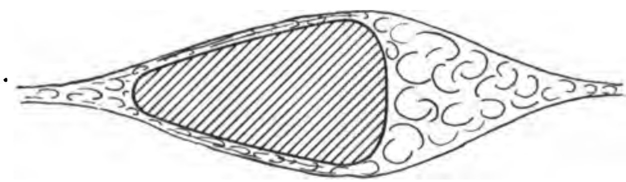

FIG. 48.

that the water at the stern will be moving in a very confused and irregular manner. Fresh portions of water will be drawn into the eddying mass, whilst other eddies will be absorbed by viscosity.

Thus, due to sharp discontinuities, there is always a dissipation of energy, the amount dissipated being dependent on the form of the body. Not only might eddies be formed at the stern, but they might be also formed at the bow; but, comparing Figs. 47, 48 it is clear that a blunt bow is less objectionable than a blunt stern. Thus, there is a second resistance, called *eddy resistance.*

§ 47. **Wave-making Resistance.**—Thus, due to surface friction and eddy-making, there is a considerable variation of pressure along any average line of flow from what would be calculated in the ideal case. But, further, in the case of a ship moving on the surface of water, there is a variation of pressure along the stream lines. It was shown (§ 11) that in the neighbourhood of the bow and stern there is an excess pressure, and, amidships, a defect pressure. Thus, especially at the bow and stern, elevations are formed which are the primary waves of a wave system. A secondary system may be formed amidships. These waves spread away from the ship and carry with them a considerable portion of their energy. The maintenance of the wave system requires, therefore, a constant supply of energy from the moving body, and this, as in previous cases, gives rise to a resistance to the motion called the *wave-making resistance.*

If it were possible to study the amount and nature of the disturbance of the forces acting on the body, the immersed surface could be imagined divided into a number of small elements. The force on each element may be resolved into a normal and tangential component and which includes all effect due to all causes. The longitudinal component of the force, integrated all over the surface, represents the "resistance."[1] Such a treatment would assume an exact knowledge of the distribution of pressure which, on account of the complicated motion of the water, it would be impossible to obtain theoretically. It has been suggested that the variation of pressure may be obtained experimentally, but the experimental difficulties are great.[2]

The only satisfactory solution is to obtain quantitative data for

[1] This was the method adopted by Professor Rankine in his article on "Augmented Surface."

[2] Herr B. Schieldorp, *Transactions of the Institution of Naval Architects*, 1898.

each resistance, without attempting any detailed analysis of the forces.

The resistance may, therefore, be divided under three heads—

(1) Eddy resistance.
(2) Skin resistance.
(3) Wave-making resistance.

Each of these has to be determined separately, and the net resistance found. It is practically assumed that these resistances act independently, and the total resistance is the sum of the three resistances. Experiment justifies this assumption. The eddy resistance and the wave-making resistance taken together are usually referred to as the *residuary* resistance.

§ **48. Theoretical Investigation on Eddy Resistance.**—Consider the case of a plane lamina (Fig. 49) placed perpendicularly to a

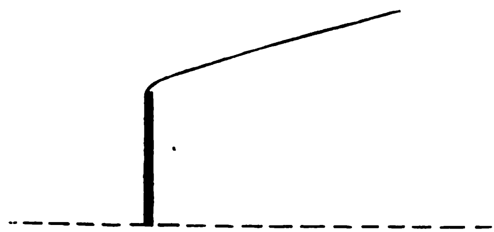

Fig. 49.

stream. The stream lines impinge perpendicularly on the plate at the centre, and glance off in a tangential direction at the edges of the plate. There will be a surface of discontinuity between the dead water behind the plate and the moving water outside; and the curve of separation will extend to infinity. For the purposes of calculation the pressure is assumed constant along this surface of discontinuity (Fig. 49), similar to a jet discharging into the atmosphere. On the front side there is an augmentation of pressure, being a maximum at the centre and diminishing to zero at the edges of the plate. At the centre the excess pressure is

$\frac{wc^2}{2g}$, c being the velocity in the uniform stream

and the resultant pressure[1] is

$$\frac{\pi}{4+\pi}\cdot\frac{w}{g}c^2l,\ l \text{ being the length of the plate}$$

$$= 0{\cdot}44\frac{w}{g}c^2l.$$

If the lamina be inclined, the same method is adopted and the total *normal* pressure[2] is

$$\frac{\pi \sin a}{4+\pi \sin a}\cdot\frac{w}{g}c^2l.$$

As before, the pressure is not uniform over the lamina. At K (Fig. 50) the excess pressure is a maximum, and equal to $\frac{w}{2g}c^2l$.

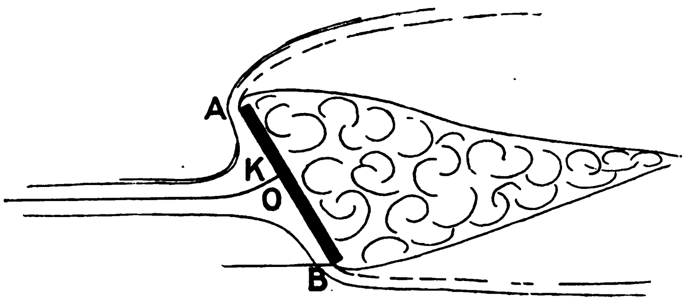

FIG. 50.

The centre of pressure is at a point given by

$$OX = x = \frac{3}{4}\cdot\frac{l\cos a}{4+\pi\sin a}$$

O being the centre of the plate.

If the plate be pivoted, then a is determined when x, the distance of the pivot from the centre of the plate, is known. Thus

$$\text{if } a = 90^\circ,\quad x = 0,\quad \text{and pressure} = 0{\cdot}44\frac{w}{g}c^2l$$

$$\text{if } a = 0,\qquad x = \frac{3}{16}b,\quad ,, \qquad ,, \qquad = 0.$$

[1] Lamb's "Hydromechanics," p. 109.
[2] *Ibid.*, pp. 110, 111.

The latter case represents a plate edgewise to the stream. If x be greater than this, the plate will set itself edgewise to the stream. This is the principle of balanced rudders.

It has been pointed out that, in actual cases, the surfaces of separation do not extend to infinity, but form dead water behind the plate. This consists of an eddying mass of water and the surface of separation is unstable, so that there is a constant interchange of water between the eddying mass and the outside water. There must, therefore, be a reduction of pressure in the rear; and additional resistance due to this is called the negative resistance.

§ 49. **Experimental Results on Eddy Resistance.**—In making experiments, care must be taken to have the plate immersed at such a depth that there is no surface disturbance, otherwise the resistance will be considerably increased. When sufficient depth has been reached, the resistance will be independent of depth. Moreover, the speed of the body must be uniform, otherwise an estimate will have to be made for virtual mass.[1]

Experiments are not very consistent, but they agree in expressing the normal pressure in the form

$$R = k\frac{w}{2g}Av^2$$

in which k is a coefficient and A the area of the plate. For salt water

$$R = kAv^2.$$

In a flat plate moving normally to the stream,[2] Beaufoy found that $k = 1{\cdot}13$, Joessel 1·62, and Mr. R. E. Froude 1·1. It is approximately the same for a square as for a circular plate. The resistance at 10 feet per second per square foot is 112 pounds.

For plates inclined to the stream at a small angle Mr. W. Froude[3] obtained—for propeller blades—

$$\text{normal pressure} = 1{\cdot}7Av^2 \sin a$$

[1] When a body moves on the surface of water the particles of water surrounding it have a certain velocity impressed upon them. This adds to the mass of the body.

[2] Sir William White's "Naval Architecture," p. 437.

[3] *Transactions of the Institution of Naval Architects*, 1878.

and Joessel[1] (for balanced rudders) obtained

$$\text{normal pressure} = 1{\cdot}62 \frac{\sin a}{0{\cdot}39 + 0{\cdot}61 \sin a} \cdot \frac{w}{g} \mathrm{A}v^2.$$

For small angles this is

$$= 4{\cdot}1 \frac{w}{2g} \mathrm{A}v^2 \sin a$$
$$= 4{\cdot}1\ \mathrm{A}v^2 \sin a.$$

The value of x is obtained from the formula

$$x = 0{\cdot}305l\,(1 - \sin a).$$

Hagen made experiments on the value of x, the results being given in the following table:—

$a =$	90	78	60	48	25	13	8	$6\frac{1}{2}$	4
$\frac{x}{l} =$	0	0·026	0·055	0·088	0·125	0·167	0·214	0·27	0·333

The experiment of Mr. W. Froude would only be true for angles less than 15°. Joessel's results are for balanced rudders. The experiments were made in the Loire, and the maximum speed was $2\frac{1}{2}$ knots, or 42 feet per second. They would probably apply from 20° to 45°. For small angles, the ratio of the coefficients as given by Mr. W. Froude and Lord Rayleigh is $\frac{1{\cdot}7}{1{\cdot}57} = 1{\cdot}08$. For 90°, the ratio of the coefficients as given by Mr. R. E. Froude and Lord Rayleigh is $\frac{1{\cdot}12}{0{\cdot}88} = 1{\cdot}27$. It appears, therefore, that the true normal pressure exceeds that by Lord Rayleigh by 8 to 27 per cent., the percentage being higher the greater the angle of inclination, the difference representing the suction pressure. As regards the value of $\frac{x}{l}$, both Joessel and Hagen give much greater values than Lord Rayleigh, but for angles of over 15°, Hagen and Lord Rayleigh practically agree. Sir William White[2] states that below 15° there

[1] *Transactions of the Institution of Naval Architects*, 1873.
[2] "Naval Architecture," p. 662.

is considerable uncertainty in the value of $\frac{l}{x}$, but above that the values are fairly well known. The results obtained by Mr. W. Froude and Joessel agree closely with one another and confirm the general accuracy of Lord Rayleigh's formula.

Approximately, when

a	10°	20°	30°	40°
$\frac{x}{l}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{8}$	$\frac{1}{10}$

The most recent attempt to solve the problem is by Mr. Calvert.[1] The experiments appear to have been carried out on plano-convex blades, for angles varying from 0° to 15°. Mr. Calvert found that the normal resistance $\propto V^{1\cdot 85}$ and not as l, but as l^m for all other angles where m is fractional, but $= 1$ when $a = 90°$. The plate tested was not isolated, but when a part of a screw-propeller blade, Mr. Calvert verified that the normal pressure varied as sin a.

FIG. 51.

The coefficients quoted refer to a flat plate. A propeller is convex at the back (Fig. 51). The probability is that the eddies extend over the whole area, instead of being localized at the back of the leading edge. The value will then probably be about 2·5 instead of 1·7.[2]

With the body of any other shape, the eddy resistance may be expressed by a formula of the kind

$$k \cdot \frac{w}{2g} A v^2$$

in which A is the sectional area of the body exposed to the stream. For a sphere k is about 0·4, and for a cylinder 0·5.

[1] *Transactions of the Institution of Naval Architects*, 1887.
[2] Cotterill's "Applied Mechanics," p. 496.

Surface Friction

When a ship moves through the water, an eddying belt is formed, which surrounds the ship, and, at the rear of the ship, the water is in an agitated condition. As already pointed out (§ 45), there is surface friction between the water and the hull of the ship, and this causes a resistance to the ship's motion.

§ **50. Froude's Experiments on Surface Friction.**—The first experiments made on surface friction were those by Beaufoy.[1] Smooth, painted planks were drawn through the water, and the resistance was measured.

The only reliable experiments on surface friction are those by Mr. William Froude.[2] These experiments were made in a tank 278 feet long, 36 feet wide at the top, the depth of the water being 8 feet 9 inches. The boards were about $\frac{3}{16}$ inch thick and 19 inches deep, the top edge being $1\frac{1}{2}$ inch below the surface. Their length varied from 1 foot 6 inches to 50 feet. In making the experiments, the boards to be tested must be quite free in the direction of motion, but constrained in all other directions. To determine a law of resistance, the *force, distance run*, and *time* must be measured.

A carriage, A (Fig. 52), runs along rails over the tank, and from it a parallel motion, BB, which allows a frame, C, to rock freely in a longitudinal direction, but prevents motion in a transverse direction. The plank is loaded with a lead keel, D, so that it floats immersed, and is provided with a cut-water, E, which extends upwards and is fastened to the hanging frame. The spring F is pivoted to a projecting piece attached to CC, and also to a stout bracket, G, attached to the carriage A. The extension of the spring F was magnified in the manner shown, a link, H, being pivoted to the carriage and to the spring F, and a second link turning on a pivot attached to the bracket G, and long enough to actuate the pencil K, which records on the rotating drum a distance proportional to the resistance. The drum is connected by gearing to the truck wheels, and an electric clock

[1] "Nautical and Hydraulic Experiments," 1834.
[2] *British Association Reports*, 1872-1874.

marks equal intervals of time. Thus a curve of resistance as a speed basis may be deduced.

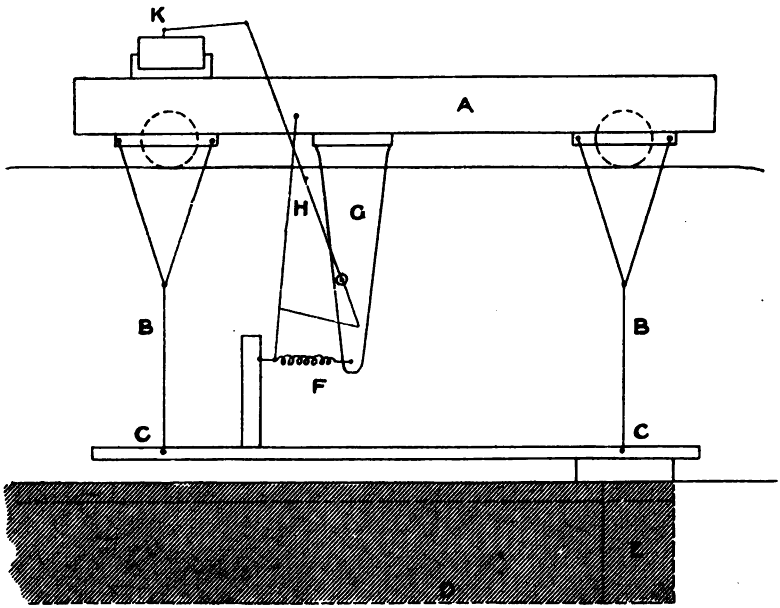

Fig. 52.

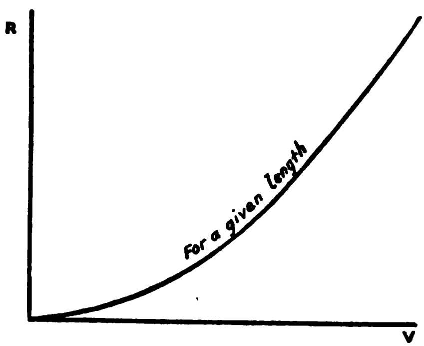

Fig. 53.

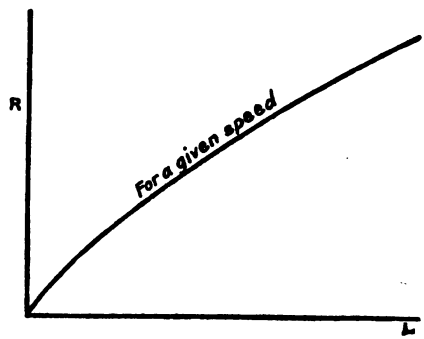

Fig. 54.

The object of the experiments was to determine the law of variation of resistance with (1) velocity, (2) length, (3) quality of surface.

The method of making the experiments was to obtain a series of *resistance-velocity* curves for each length of board (Fig. 53). From these a set of *resistance-length* curves for different velocities could be deduced (Fig. 54). In the first set of experiments the latter curves did not pass through the origin, but above it, showing that the resistance was not zero for zero length. To plot the curve in the neighbourhood of the origin, experiments had to be made on lengths smaller than 1 foot 6 inches. At first, the bow of the plank was rounded off, whilst the stern was blunt. On shaping the bow to an edge, the resistance for $l = 0$ was made slightly smaller; but on shaping the stern to an edge it practically disappeared. The other corrections that had to be made were—(1) the air-resistance of the swing bar, (2) the resistance due to the excess surface of the cut-water over the plank proper, (3) any irregularities in the surface caused by the connections. Of these, the first was determined by direct experiment; the second could be determined from data already possessed; whilst the third was obtained by running the 18-inch length smoothed up with paraffine and varnished.

When these corrections were made, it was found that the curves giving the relation between resistance and length passed through the origin, thus showing that the effect was all surface friction.

It was found that the latter curves were not straight, but at all speeds were concave to the base line. This showed that the resistance did not vary as the length, but at a less rate. This result is due to the portion of surface which, moving first in the line of motion, by its resistance on the water, must communicate to the water motion in the same direction; and, consequently, the portion of the surface which succeeds the first will be rubbing, not against stationary water, but against water partially moving in its own direction, and cannot, therefore, cause as much resistance from it. The average forward velocity of the eddying mass will increase as the water approaches the stern.

§ 51. **Experimental Results.**—Mr. Froude found that for a given plank the resistance varies as v^n, where n was constant for a particular plank, but, in different planks, depended on the length and quality of the surface. In any particular case, n can be found by logarithmic plotting, the criterion of constancy of n being that

we get a straight line on the log chart. Mr. Froude expresesd his in the form of a table, in which the speed was 10 feet per second, and the experiments were made in fresh water. He experimented on lengths of 2 feet, 8 feet, 20 feet, and 50 feet; and surfaces of varnish, paraffine, tinfoil, calico, fine sand, medium sand, and coarse sand.

His results are embodied in the following table:—

	Fresh water: speed 10 feet per second.											
	2 feet.			**8 feet.**			**20 feet.**			**50 feet.**		
	n	Mean resistance per sq. ft. over whole length.	Value of resistance in pounds over last sq. ft.	n	Mean resistance per sq. ft. over whole length.	Value of resistance in pounds over last sq. ft.	n	Mean resistance per sq. ft. over whole length.	Value of resistance in pounds over last sq. ft.	n	Mean resistance per sq. ft. over whole length.	Value of resistance in pounds over last sq. ft.
Varnish	2·0	0·41	0·39	1·85	0·325	0·264	1·85	0·278	0·240	1·83	0·250	0·226
Paraffine	1·95	0·38	0·37	1·94	0·314	0·260	1·93	0·271	0·237			
Tinfoil	2·16	0·30	0·295	1·99	0·278	0·263	1·90	0·262	0·244	1·83	0·246	0·232
Calico	1·93	0·87	0·725	1·92	0·626	0·504	1·89	0·531	0·447	1·87	0·474	0·423
Fine sand	2·00	0·81	0·690	2·00	0·583	0·450	2·00	0·480	0·384	2·06	0·405	0·337
Medium sand	2·00	0·90	0·730	2·00	0·625	0·488	2·00	0·534	0·465	2·00	0·488	0·456
Coarse sand	2·00	1·10	0·880	2·00	0·714	0·520	2·00	0·588	0·490			

Very approximately, the results may be expressed by the formula

$$R = fSV^n.$$

The value of f depends on—

(1) The quality of the surface.

(2) On the length of the surface, generally, not always, falling off as the length increases.

(3) Slightly, but very slightly, on temperature, being less the higher the temperature.

(4) Entirely independent of pressure.

(5) Proportional to the density of the fluid. This is due to the fact that it is an eddy resistance, and has been verified for pipes. The resistance in salt water is thus $2\frac{1}{2}$ per cent. greater than fresh water.

The value of n depends in some way on the quality of the surface, and generally, not always, decreases with length.

It will be noticed that if f and n are not independent, and if n falls off rapidly (as in tinfoil) without a corresponding rapid falling off in R, then f will increase.

In Fig. 55, the abscissæ are log V, and the ordinates log R. Five

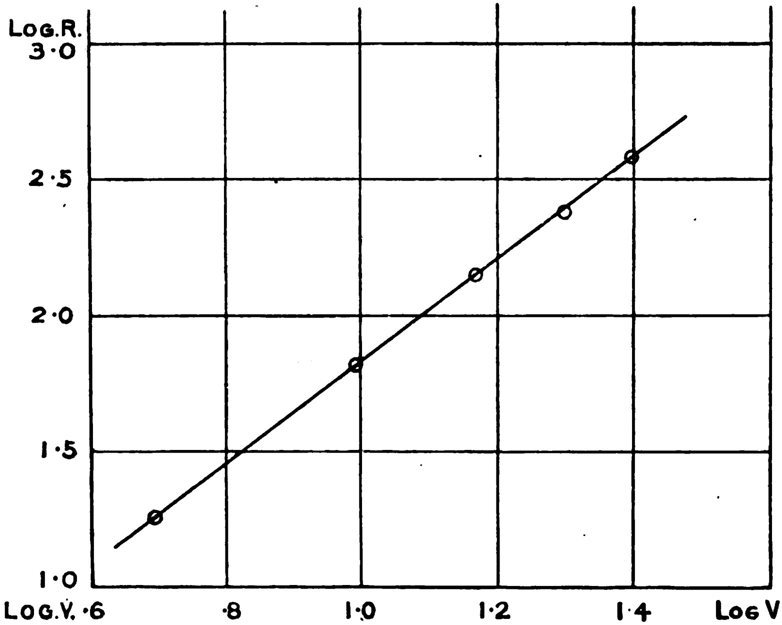

FIG. 55.

points are plotted, corresponding to 5, 10, 15, 20, 25 feet per second for a 50-feet length. The points are marked with a circle. It will be noticed that the points are almost exactly on a straight line. Taking the two end points, which are exactly on a straight line, and reading the scales

$$n = \frac{2{\cdot}558 - 1{\cdot}279}{1{\cdot}398 - 0{\cdot}699} = \frac{1{\cdot}279}{0{\cdot}699} = 1{\cdot}83.$$

To obtain the value of n and f, let

$$R = fSV^n$$

then $$\log R = \log fS + n \log V$$

and $$\therefore \log \frac{R_1}{R_2} = n \log \frac{V_1}{V_2}$$

$$n = \frac{\log R_1 - \log R_2}{\log V_1 - \log V_2}.$$

The values of n and f, deduced from the curves, are given in the following table:—

Length.	2		8		20		50	
	n	f	n	f	n	f	n	f
Varnish . . .	2·00	0·0041	1·85	0·0046	1·85	0·00393	1·83	0·0037
Paraffine . . .	1·95	0·00426	1·94	0·0036	1·93	0·00318		
Tinfoil . . .	2·16	0·00208	1·99	0·00284	1·90	0·00331	1·83	0·00364
Calico	1·93	0·0102	1·92	0·00754	1·89	0·00685	1·87	0·00639
Fine sand. . .	2·00	0·0090	2·00	0·00625	2·0	0·00534	2·00	0·00488
Coarse sand . .	2·00	0·0110	2·00	0·00714	2·0	0·00588		

The earlier theories of resistance assumed that the resistance varied as the square of the velocity, but the correct index is 1·83. This difference, although small, causes a reduction of 32 per cent. at 10 knots, nearly 40 per cent. at 20 knots, and 42 per cent. at 25 knots.

§ **52. Froude's Method of Extension to Actual Ships.**—Mr. Froude's experiments were made on planks having a maximum speed of 800 feet per minute, that is to say, about 8 knots. In extending them, therefore, to full-sized ships, certain assumptions have to be made. The usual assumption to make is that the coefficient of resistance is independent of speed, and to compute the resistance by assuming that the mean resistance over the first 50 feet is that given by him, and that over the remainder it has a mean resistance equal to that over the fiftieth foot. This would probably slightly exaggerate the resistance.

Mr. W. Froude, by experimenting on models, came to the conclusion that the immersed skin is equivalent to that of a rectangular surface of the same area and of the same length, in the line of motion, as that of the model. The values of f and n are deduced from the experiments on planks (§ 51).

Mr. W. Froude, at Torquay, used a paraffine shell. Now, Mr. R. E. Froude, at the Admiralty tank at Haslar, uses a varnished model. The value of n is, according to Mr. W. Froude, higher than that for a varnished surface, and the coefficient f is somewhat different, as the table shows. Mr. R. E. Froude, however, now finds, that whatever the explanation—possibly due to a different quality of paraffine—both the coefficient and exponent are substantially the same as for varnish. Experiments are now made on all models with both surfaces.

A table from which the values of f and n in the formula

$$R = fSV^n$$

may be found, is given by Mr. R. E. Froude, and is as follows:—

L in feet . . .	50	100	200	300	400	500	600
n	1·825	1·825	1·825	1·825	1·825	1·825	1·825
f (speed in feet per second) .	0·00371	0·00355	0·00347	0·00344	0·00342	0·00339	0·00337
f (speed in knots)	0·00963	0·00913	0·00902	0·00892	0·00886	0·00880	0·00874

§ 53. **Estimation of Wetted Surface of a Ship.**—The only satisfactory way to determine the wetted surface of a ship, is, from the drawings of the ship, to estimate it by dividing the ship's hull into a number of horizontal compartments, and estimating each separately.

But this is not always possible. It might be required to predict, approximately, the wetted surface from other examples of a near type. There are many formulæ designed to give the wetted surface, a few of which will be noticed.

Let S = wetted surface in square feet;
L = length in feet;
B = beam;
D = mean draught (excluding bilge keels);
Δ = displacement in tons.

Then 35Δ is the actual immersed volume in cubic feet, BDL is

the immersed volume of a block of the same extreme dimensions as the ship. The ratio $\frac{35\Delta}{BDL}$ is called the *block coefficient*, or *coefficient* of fineness, and is denoted by β.

A rough approximation is to compute the surface by taking the area of the bottom and the two sides, giving

$$S = 2DL + \frac{35B}{D}$$

$$= L(2D + \Delta B).$$

Mr. Denny finds that a nearer approximation is

$$= L(1{\cdot}7D + \beta B).$$

From experiments made at the Haslar tank, a formula frequently employed is

$$S = \nabla^{\frac{2}{3}}\left(3.4 + \frac{L}{2\nabla^{\frac{1}{3}}}\right)$$

which ∇ is the displacement in tons, 1 ton of salt water is 35Δ cubic feet. This no doubt refers principally to Admiralty types, whilst Mr. Denny's will refer to types in the mercantile marine.

§ **54**. **Frictional Wake.**—The effect of frictional resistance is to cause a frictional wake, that is to say, a general motion of a certain body of water immediately behind the plank. The forward motion of the eddying belt of water will increase in width as the stern is approached, after which it will remain constant. The wake will be lengthened at a constant rate. Thus

The resistance = the momentum generated per unit time, notwithstanding the presence of eddies which, although representing a considerable amount of energy, have zero momentum.

Let V = velocity of plank;

v = final velocity impressed upon a layer of water of thickness dh and of unit depth.

The rate at which the wake is lengthened is $V - v$, and the mass per second which receives motion is

$$\int(V - v)\frac{w}{g}dh.$$

The final momentum of the wake is

$$\int \frac{w}{g}(V - v)v dh$$

the integral extending over the whole width of wake.

To obtain quantitative results some assumption must be made. Frequently, the velocity of the wake is taken to decrease uniformly from V at the central strip of the wake to zero at either side. Thus

$$v = \frac{H - h}{H} . V$$

$$\text{therefore resistance} = \int \frac{wV^2}{gH^2}(Hh - h^2)dh.$$

If R be the resistance for one side of the plank, the limits of the integral are H and 0, and

$$R = \frac{w}{6g} V^2 H = \tfrac{1}{3}V^2 H \text{ for salt water.}$$

The distribution of velocity in a sternward direction as well as in a transverse direction will no doubt be considerably affected by the eddies. The only experimental work on the subject is by Mr. Calvert,[1] who towed a plank on the surface of the water. At different points underneath the plank, tubes were led to gauge glasses, the heights of the water in them being proportional to the square of the velocity with which the tube moved through the surrounding water. Simultaneous readings could be taken by photographing the gauges. The tubes were $\frac{1}{8}$ inch internal diameter. At distances of 1, 7, 14, 21, and 28 feet from the leading edge, the forward speeds of the wake were 16, 37, 45, 48, and 50 per cent. of the wake; and these proportions appeared to be maintained at all speeds between 200 and 400 feet per minute. It will be noticed that the velocity of the forward wake increases rapidly near the leading edge, but becomes sensibly constant for lengths of 50 feet. Mr. Calvert also fitted wire frames to the plank, and fixed the tubes at different depths. He found that the velocity decreased in a geometrical ratio as the distance from the surface increased by equal amounts.

[1] *Transactions of the Institution of Naval Architects*, 1893.

These experiments show that the forward velocity of the layer near the plank practically became constant when the length exceeded a certain quantity, and that the increased momentum consequent upon increased resistance with greater length was obtained by the mean speed of the wake being increased as the stern was approached. In any case, the plank is surrounded by a favouring wake, and Mr. Froude—in his paper on skin resistance—expressed surprise that the reduction due to length was not greater than experiment showed it to be.

§ 55. **Analysis of the Motion of the Water in the Wake.**—Professor Cotterill,[1] discussing the motion of the water in the wake, takes the case of a board moving edgewise through the water, between two parallel plates placed horizontally. Following the board, there is a narrow deep current which may be represented by an "equivalent uniform wake" at the rear of the board. Now, the flow across any section, either in front or behind the board, must be zero, and consequently, in order to counteract the forward flow in the water, there must be a counter-current in the opposite direction. Flow can only be set up in this counter-current provided there is a drop in pressure between the front and rear of the board, so that there must be a drop in pressure, or, if the upper plate be removed, a slope of surface from front to rear. The slope will not be uniform across the stream, but will have a mean value. It is in this way that the velocity of the water gliding over the board is prevented from diminishing indefinitely when the length of the body is increased. Apparently, the resistance produces the wake, which in turn causes a reduction of pressure in the rear, and, therefore, tends to accelerate the water. On the outer streams, the velocity induced will be merely that due to the difference in pressure; but near the plank this will not be the case. Now, in the case of an indefinitely thin plank, the resistance must always be equal to the momentum generated per second in the wake, because, although the difference exists, it produces no effect on the resistance. Mr. Froude's experiments measure the former, but not the latter resistance. In an actual ship, not only will the former resistance have an effect, but so also will the

[1] *Transactions of the Institution of Naval Architects*, 1887.

latter resistance. This extra resistance—or, as Professor Rankine terms it, the indirect resistance—will be proportional to

$$\text{midship section} \times \text{drop in pressure}$$

or

$$\text{displacement} \times \text{slope per unit length.}$$

Wave-making

When a vessel moves on the surface of the water, waves are produced on the free surface. These waves follow their natural tendency to spread, and much of the energy required for their creation is lost to the ship. The energy necessary for their creation and continuance must be supplied by the source of energy within the ship itself, and constitutes an additional resistance to the ship's motion.

The trochoidal wave has been fully discussed in Chapter II. All ocean waves are of this type. If a velocity equal to that of propagation be impressed upon the whole system, the wave becomes a stationary wave, and all the particles flow in trochoidal paths.

Ships' waves are more complicated than the trochoidal wave; but, whatever the system, it may be taken as stream line flow past a ship-shaped body.

§ 56. **Law of Comparison in Ships.**—Take the case of a ship floating on the water. When the ship moves through the water a wave system is formed; if the ship be imagined to be brought to rest, by impressing a velocity equal to that of the ship on the whole system, then the water will flow past the ship in stream line motion, and the surface stream line will follow the undulations of the surface.

Take the case of a ship, imagined stationary, with the water flowing past it. And suppose a small model, exactly similar in shape to the ship, also floats on the water. At a certain speed, the ship will form a particular system of stream lines. Now, suppose the water flows past the model. At any arbitrary speed, the stream line system will be quite different from that of the ship; but the speed may be increased so that—since the ship and model

are in every respect similar, and hydraulic phenomena are independent of dimensions—the stream line system of the model is exactly similar and similarly situated to that of the ship. The ratio of corresponding speeds will depend on the ratio of the mean dimensions.

Fig. 56. Imagine a ship and model of similar shape. The stream

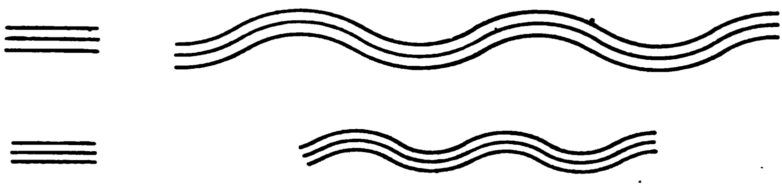

FIG. 56.

lines will be as shown. Consider two corresponding stream tubes. Let large letters refer to the ship, and small letters refer to the model. Let symbols without suffixes refer to any pair of arbitrary corresponding points, and symbols with suffix *o* to a particular pair of corresponding points. Let W, w be weights per cubic foot of salt water and fresh water, so that the ratio of the densities is $\frac{64}{62 \cdot 5} = 1 \cdot 0625$.

From the energy equation

$$\frac{P}{W} + \frac{V^2}{2g} + Z = \frac{P_o}{W} + \frac{V_o^2}{2g} + Z_o$$

$$\frac{p}{w} + \frac{v^2}{2g} + z = \frac{p_o}{w} + \frac{v_o^2}{2g} + z_o.$$

Also $$Z = nz, \quad Z_o = nz_o$$

$$\left(\frac{P}{W} - n\frac{p}{w}\right) - \left(\frac{P_o}{W} - n\frac{p_o}{w}\right) + \frac{V^2 - nv^2}{2g} - \frac{V_o^2 - nv_o^2}{2g} = 0.$$

Also, equating the flow along the tubes,

$$AV = A_oV_o$$

$$av = a_ov_o$$

$$\therefore \frac{V}{V_o} = \frac{v}{v_o}.$$

Thus the velocity ratio at every pair of corresponding sections is the same. The above equation becomes

$$\left(\frac{P - P_o}{W} - n\frac{p - p_o}{w}\right) + \frac{\left(1 - \frac{v_o^2}{v^2}\right)}{2g}(V^2 - nv^2) = 0.$$

The equation is true for any pair of corresponding stream tubes; in particular, for the surface tube. For this particular stream tube, the surface pressure is constant; so that whatever the absolute pressure of the atmosphere, or whether it be the same in the two cases,

$$p = p_o, \qquad P = P_o$$

and, therefore, $$V^2 = nv^2.$$

Thus, for the surface stream tube, the velocity ratio at any pair of corresponding points varies as $\sqrt{n}$.

Next, consider the stream tube immediately below the surface stream tube. The pressure varies in a direction normal to the stream line for two reasons, namely, (1) due to gravity, (2) due to centripetal acceleration necessary to cause the curvature of the streams. Whether the atmospheric pressures be the same or not, due to gravity, the change of pressure is proportional to n, if the fluids be of equal density; or to $n\frac{W}{w}$ if they are of different density. Again, if $d\sigma$ represent a small volume of water at a point in the subsurface stream line, ρ the radius of curvature of the stream line, and v the velocity of a particle of water, then the change of force due to centripetal acceleration is proportional to

$$\frac{w}{g}\cdot\frac{v^2}{\rho}\cdot d\sigma.$$

The change of intensity of pressure is, therefore, proportional to

$$\frac{wv^2}{g\rho}\cdot\frac{d\sigma}{da}$$

where da is an element of area. Thus the ratio of the increments in pressure per square inch is

$$\frac{WV^2}{wv^2}$$

and of pressure heads, above atmospheres,

$$\frac{V^2}{v^2} = n.$$

Thus, in the subsurface stream tube, the velocity ratio is equal to the $\sqrt{n}$. Hence, from the last equation, at every pair of corresponding points,

$$\frac{P - P_o}{W} = n\frac{p - p_o}{w}.$$

Expressed otherwise, the difference of pressure head in the subsurface streams is in the ratio of n. The principle may be extended through the mass of liquid. At a great distance from the ship the pressures P_o, p_o are static pressures, and at corresponding points,

$$\frac{P_o}{W} = n\frac{p_o}{w}.$$

Thus at *all* corresponding points the ratio of pressure heads varies as n, and the ratio of velocities as $\sqrt{n}$.

Consider a ship and a model of exactly similar proportions. Imagine the ship to move through the water, say salt water, and the model to be towed through fresh water. If the ship and model be run at corresponding speeds, then the wave systems will be geometrically similar. In the wave system the variation of pressures is different from that in water at rest, and at each point in the wave system the pressures can be determined (Chapter II.). Imagine two elemental areas of the hull, which are similar and similarly situated. Let P, p be intensities of pressure over those areas, and let θ be the inclination of the normal to the fore and aft plane. The resistance caused by this small element is the resolved part of the pressure in a fore and aft direction, and therefore if dS, ds represent the two elemental areas—

$$\text{the resistance ratio} \quad = \frac{PdS \cos\theta}{pds \cos\theta} = n \frac{W}{w} n^2$$

$$= n^3 \frac{W}{w}$$

$$= \frac{\text{cubic displacement of ship}}{\text{cubic displacement of model}} \cdot \frac{W}{w}$$

$$= \frac{\text{displacement of ship in tons}}{\text{displacement of model in tons}}.$$

This applies to all corresponding elemental areas.

An alternative proof is to consider the energy in the wave system. In § 32 it has been proved that the rate at which energy is drained away from the ship is half the energy of the wave system; the energy of a trochoidal system has been shown to be

$$\frac{wlh^2b}{8} \text{ approximately.}$$

Thus the energy draining away varies at the fourth power of the linear dimensions. Now, this energy must be equal to the products of the resistance experienced by the ship or model, multiplied by the distance run; hence the

$$\text{resistance ratio} \quad = n^3 \frac{W}{w} \text{ as before.}$$

§ 57. **Law of Comparison.**—The law of comparison, applied to ships and models, is—

Resistances are proportional to the tonnage displacements, at speeds proportional to the square root of the linear dimensions; that is, to the sixth root of the cubic dimensions or displacements.

For example, if the surface of water surrounding a ship 160 feet long, at 10 knots, were modelled together with the ship, on any scale, the model would equally represent, on half that scale, the water surface surrounding a ship of similar form 320 feet long, travelling at 14·14 knots; or, again, on sixteen times that scale, the water surface surrounding a model of the ship 10 feet long, travelling at $2\frac{1}{2}$ knots. The resistances in the three cases would be as

$$1 : 8 : \frac{1}{4096}$$

the density of the fluid being the same in each.

The above law assumes that the true stream line motion is unbroken. It neglects the formation of eddies which may be due either to skin friction or to discontinuities in form. The existence of these eddies will obviously modify the pressures acting on the hull, and so will modify the variation of pressure and velocity along a stream line, or, say, along an average line of flow. If two ships are geometrically similar, and run at corresponding speed, probably the presence of eddies will not disturb the similarity of the wave system. The eddying water might, in other words, alter the virtual mass of the ship, but in a similar manner. This statement does not necessarily imply that the law of comparison is applicable to skin friction or eddy making.

Experiment, however, justifies the assumption that, notwithstanding the disturbing of eddy and skin resistance, the law of comparison as regards wave making is true. The confirmation is derived from experiments of different-sized models—some being of very peculiar shape—and also by comparing the resistance of a ship, as computed from the model, with that actually obtained on a full-sized ship.

§ 58. **Method of reducing the Results.**—Mr. W. Froude made an extensive series of experiments on models in order to establish relations between the proportions of a ship on the wave making resistance at different speeds.

The experiments were made in the Admiralty tank at Torquay, now removed to Haslar, under the superintendence of Mr. R. E. Froude. The length of the tank was 400 feet, the width 21 feet, and the depth 9·5 feet. The models were 10 to 28 feet long, usually 14 feet, and were made of paraffine. The speeds usually obtained varied from 100 to 500 feet per minute, with a maximum of 1200 to 1400 feet per minute, so that the minimum time of traversing the tank—from start to finish—was 20 seconds. The paraffine shells were usually one inch thick, and were shaped by a special machine.

The system is that of determining the scale of resistance of a model of any given form, and from that deducing the resistance of a ship of similar form, rather than that of searching for the best form—the form which is best adopted to any given circumstances

coming out incidentally from a comparison of the various results. Each model is driven through the water at the successive assigned appropriate speeds by an extremely sensitive dynamometrical apparatus, which gives, in every case, an accurate autographic

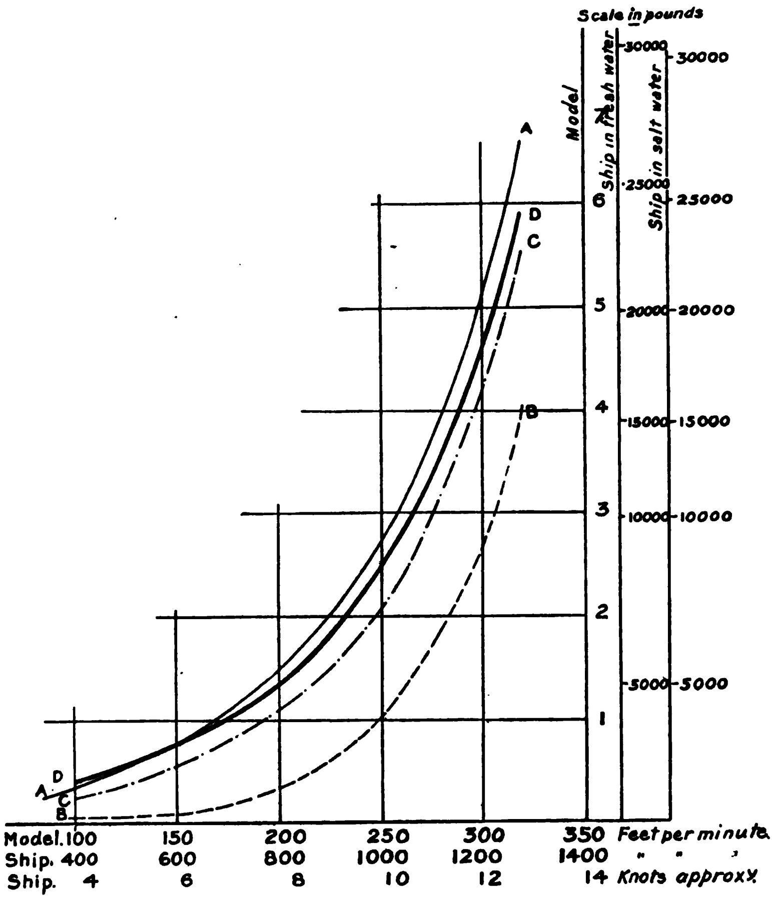

FIG. 57.

record of the speed and resistance. The description of the apparatus has already been given in § 50, Chapter III., the model being substituted for the plank.

The total resistance of the model at a series of speeds is thus

obtained, and a curve of resistance on a speed base can be at once plotted. This includes surface resistance, eddy resistance, and wave-making resistance. The two latter follow the law of comparison, and may therefore be called the *residuary resistance;* the former does not follow the law of comparison, and must be considered separately, both for the model and ship. Thus, Fig. 57, AA represents the total curve of resistance for the model. The proper values of f and n must be taken according to its length and surface (§ 51), and the skin resistance calculated at different speeds. It is convenient to plot the calculated resistance by setting down from the curve AA, giving the curve BB. The ordinates B, B then represent the residuary resistances of the model; and by allowing the scale of ordinates in proportion to the displacements, and the scale of abscissæ in proportion to the square root of the linear dimensions, the curve BB may be considered the curve of residuary resistance of the ship.

Knowing the wetted surface and length of the ship, taking the proper values of n (§ 52), the frictional resistance at different speeds can be calculated and, by plotting these above BB, the curve CC, which, on the new scale, represents the total resistance of the ship.

The diagram (Fig. 57) refers to the *Greyhound*.[1] The model was one-sixteenth the length of the ship, so that the resistance has to be increased in the ratio of 4096, and the speed in the ratio of 4.

A scale is given for fresh water and one for salt water. According to the theory on which the law of comparison is based, the corresponding speeds will be the same for either salt water or fresh water, so that the speed scale is the same for each. But the resistance due to wave making will be, at any given speed, proportional to the density; so, likewise, is the skin resistance.

Referring to Fig. 57, in the model at 100, 200, 300 feet per minute, the skin resistance is 89, 78, and 47 per cent. of the total; in the ship, at the corresponding speeds, it is 84, 68, and 36 per cent.

§ 59. Experiments by Colonel English.[2]—A tank, such as the

[1] *Transactions of the Institution of Naval Architects*, Mr. W. Froude, 1874.
[2] *Proceedings of the Institution of Mechanical Engineers*, Jan. 31, 1896.

Admiralty tank, with its expensive and delicate apparatus, is beyond the resources of a shipbuilding yard. In ordinary cases, the sea-going (trial) performance of a certain ship is usually known; and Colonel English gives a method of determining the indicated horse-power of a similar but larger ship, or a dissimilar ship, when running at a different speed.

If a series of progressive trials of one of the ships has been made, a curve of indicated horse-power may be plotted on a speed base. The indicated horse-power is much greater than the "effective horse-power" actually required to propel the ship, principally on account of mechanical friction in the propelling machinery, and particularly on account of loss in the screw. By assuming a "coefficient of propulsion," the effective horse-power expended in propelling the ship may be calculated. If the ships are of the same type, and not dissimilar in size, the assumption is probably correct.

Let the displacement of the first ship be Δ_1 tons, and the total resistance, as thus computed, at speed V_1, be R_1. Of this total resistance, the skin resistance, ${}_sR_1$, can be calculated, and so the wave-making resistance, ${}_wR_1$ deduced.

Let the displacement of the second ship be Δ_2, and suppose the total resistance of the second ship is required at speed V_2. The skin resistance, ${}_sR_2$, can be calculated, but ${}_wR_2$ is not known. To estimate ${}_wR_2$, an exact model is made of the first ship of any convenient displacement δ_1; so that the corresponding speed is

$$v_1 = V_1\left(\frac{\delta_1}{\Delta_1}\right)^{\frac{1}{6}}.$$

Let an exact model of the second ship be made of displacement δ_2, such that the speed v_1 is the speed corresponding to V_2 in the second ship; so that

$$\delta_2 = \Delta_2\left(\frac{v_1}{V_2}\right)^6 = \left(\frac{V_1}{V_2}\right)^6 \frac{\Delta_2}{\Delta_1}\cdot\delta_1.$$

Suppose these two models are attached to the end of a lever, the lever being of sufficient length as to prevent any interference of the wave systems of the two models, and towed through the water at speed v_1 by a rope attached to some fulcrum, the relative

lengths of the lever arms being adjusted by trial until the models tow steadily abreast. The resistance of the models are then inversely as the length of the lever arms. Let the total resistance of the second model be n times that of the first. Then

$$_wr_2 + {}_sr_2 = n({}_wr_1 + {}_sr_1).$$

Now, $_sr_2$ and $_sr_1$ can be calculated,

and
$$_wr_1 = {}_wR_1\frac{\delta_1}{\Delta_1}$$

$$_wr_2 = {}_wR_2\frac{\delta_2}{\Delta_2} = {}_wR_2\left(\frac{\delta_1}{\Delta_1}\right)\left(\frac{V_1}{V_2}\right)^6$$

whence

$$\begin{aligned}{}_wR_2 &= \frac{\Delta_1}{\delta_1}\left(\frac{V_2}{V_1}\right)^6{}_wr_2 = \frac{\Delta_1}{\delta_1}\left(\frac{V_2}{V_1}\right)^6\{n{}_wr_1 + n{}_sr_1 - {}_sr_2\}\\ &= \frac{\Delta_1}{\delta_1}\left(\frac{V_2}{V_1}\right)^6\left\{n{}_wR_1\frac{\delta_1}{\Delta_1} + n{}_sr_1 - {}_sr_2\right\}\\ &= \left(\frac{V_2}{V_1}\right)^6\left\{n(R_1 - {}_sR_1) + \frac{\Delta_1}{\delta_1}(n{}_sr_1 - {}_sr_2\right\}\end{aligned}$$

whence $R_2 = {}_sR_2 + {}_wR_2$.

The advantage of the method is that a sensitive dynamometrical apparatus is not required. The ratio only of the resistances is required, not their absolute values. The method might be extended by having a standard model whose total resistance at various speeds is known. The total resistance of the second model is then given by

$$r_2 = {}_wr_2 + {}_sr_2 = nr_1$$

$$\therefore {}_wr_2 = nr_1 - {}_sr_2$$

$$\therefore {}_wR_2 = \frac{\Delta_2}{\delta_2}(nr_1 - {}_sr_2)$$

$$R_2 = {}_sR_2 + \frac{\Delta_2}{\delta_2}(nr_1 - {}_sr_2).$$

Colonel English found that the position of the fulcrum could be found to within 0·2 per cent. of the lever length, and that the range of speed could be obtained within limits of 0·02 knot.

§ 60. Experiments on Full-sized Vessel.—The first and most important experiments made on a full-sized ship were those carried out by Mr. W. Froude, on the *Greyhound*.[1] The primary object of the experiments was to verify the law of comparison, but many other points were settled in the course of the trials. In making such experiments, it is very undesirable to take the indicated horse-power as a measure of resistance, because this includes the efficiency of the driving mechanism and of the propeller. To get definite results, the ship must be towed by another ship, and the speed, force exerted, and time automatically measured.

(1) *Measurement of speed.*—The speed was registered by paying out a continuous length of twine attached to a logship consisting of a board $2\frac{1}{2}$ feet square, and ballasted to sink 4 feet, and to set itself square to the motion. As the twine ran out it caused a drum to rotate, and the angle turned through by the drum was, on some scale, proportional to the length of twine run out. The time, also, was marked on the rotating drum, at equal intervals of time. If the logships were stationary, this would be the same as the distance run through by the ship. To eliminate the effect of the wake, the line was run out over a 20-feet spar. Moreover, to prevent the line over-running the demand of the log, and then becoming retarded by friction, a brake had to be applied which kept the tension in the twine practically constant, and equal to 2 pounds. This caused a slip at the rate of about 0·3 knot. The twine was saturated with tallow, which improved its buoyancy and prevented contraction when wet.

In measuring the distance run by the ship in this way, it must be remembered that the logship might be in a current; or, if this be not so, the ship itself might get into water having a different speed to that in which the log happens to be. In such a case, the speed of the ship relative to the surrounding water might be greater or less than that indicated by the log. According to Mr. W. Froude, the error due to this cause was the most serious one in the experiments. A check, however, on the log was given by the *Active's* screw, the number of revolutions, within wide

[1] *Transactions of the Institution of Naval Architects*, 1874.

limits, very correctly indicating the speed, since the slip varies little.

(2) *Measurement of towing force.*—The towing force was measured by a dynamometer which was practically the reverse of the hydraulic press. The full force was brought on a piston of 14 inches diameter, and the magnitude of the pressure measured by means of the deflection of a spring attached to a piston of $1\frac{1}{8}$ inch diameter. This deflection was reproduced on an enlarged scale on the recording drum previously mentioned, the recording apparatus being similar to that used on the plank experiments. The force to be measured was, of course, not the total tension in the towrope, but merely the horizontal component of that tension. This was effected by having the rope attached to a truck resting on a small length of railway on the *Greyhound's* deck, the after end of the truck being linked to the dynamometer. To reduce friction, cup leathers were dispensed with and the leakage prevented by a good fit, the fluid used in the press being oil. Frictional adhesion could be overcome by a smart blow. When the cylinder was empty, the frictional force necessary to move the piston was about 30 to 40 pounds. When the tow-rope strain was 25,000–30,000 pounds, it was 150–200 pounds. Arrangements were made to relieve the dynamometer of strain when required.

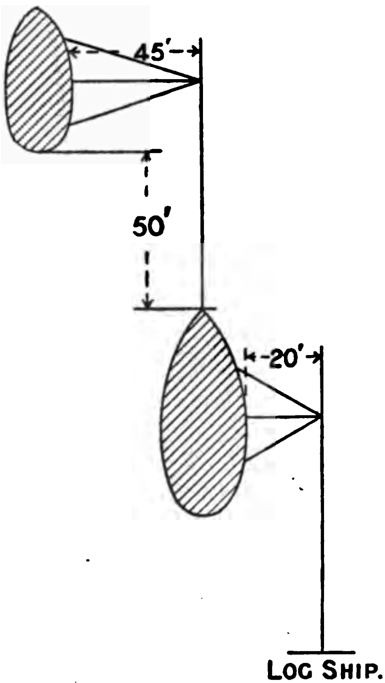

FIG. 58.

The arrangements of towing were as shown in Fig. 58, in order to avoid the wake effect of the towing ship. Had this device not been adopted, a long tow-rope would have been required, and the

alternate tightening and slackening which would have resulted might have caused some trouble. No trouble was experienced in keeping the *Active* on a straight course.

(3) *Wind effect.*—As the object of the experiments was to obtain the water resistance, the air resistance must obviously be eliminated. The velocity and direction of the wind relative to the ship were determined, in each experiment, by means of wind gauges. In order to determine the amount of air resistance, the tow-rope strain was measured when the ship moved with the same velocity, and also when the velocity of the wind varied. By assuming a law of wind resistance with velocity, the necessary coefficient was obtained. Thus when the ship moves at the same speed, let the relative velocity of the wind resolved contrary to the direction of motion be v_1 and v_2. Let M, N be the measured tow-rope strains, and let A be a wind constant for this particular ship. Then the water resistance of the ship is equal to

$$M - Av_1^2, \text{ or } N - Av_2^2$$

when A is determined.

Another method is to allow the ship to run before the wind until a steady speed is obtained. The water resistance at that speed must, of course, equal that of the wind, and, so, if the resistance has been obtained in still water, the latter can be deduced. By assuming the wind resistance to vary as the square of the speed, we could then deduce its value in any particular case.

In the case of the *Greyhound*, Mr. W. Froude found that when the relative velocity was 15 knots, the resistance was 330 pounds. At 10 knots this would give 147 pounds, and, at this speed, the water resistance was 10,420 pounds, so that the wind resistance, in calm weather, was 1·4 per cent. of the water resistance at 10 knots. Had the water resistance varied as the square of the speed, this would remain constant at all speeds provided the weather is calm. Usually, except for low speeds, the water resistance varies at a higher power than the second, so that the percentage would fall off with increasing velocity. But suppose that, when the ship was going at 10 knots, it met a 30-knot wind the percentage would be $1·4 \times 16 = 22·4$. This illustrates how much the resistance is increased when going against head winds.

The data given above assumes no rigging. With rigging, the resistance would be doubled. It refers only to the *Greyhound.*

(4) *Acceleration or retardation effect.*—When the ship is being towed, the total registered strain on the rope

= resistance of the water

+ or − the force necessary to accelerate or retard the mass of the ship and the water surrounding her.

If the speed is uniform, the second term on the right-hand side is zero. Any variation of speed would be at once shown from the automatic record, and from that record the acceleration or retardation can be calculated and the resistance of ship estimated. If the velocity of the ship increase, so also, on account of stream line action, does the velocity of every part of the system, and the force thus necessary must be derived from the ship. The virtual mass of the ship is, in other words, greater than its actual mass.

In simple cases the virtual mass can be calculated, but in a ship it must be determined experimentally. This was done by slipping the tow rope when the ship was going at a high rate of speed, and observing from the automatic record the rate at which the speed of the ship was destroyed by her resistance. Thus, for example, from the logship the clock gives a curve of *travel time;* by graphic differentiation a curve of *speed time* can be deduced; a further graphic differentiation gives a curve of *retardation time.* Thus a curve of actual retardation of a speed base can be deduced.

At any speed let the water resistance be R (previously observed), and let f' be the observed retardation at that speed. Let m_s be the mass of the ship, and m_w the mass of the accompanying water which is to be determined. Then

$$(m_w + m_s)f' = \mathrm{R}$$

from which m_w is deduced. Or, let f be the retardation on the assumption that the mass of the ship alone need be taken into account. Then

$$m_s f = \mathrm{R}$$

so that f can be calculated. Also—

$$\frac{m_w}{m_s} = \frac{f - f'}{f'}.$$

The curves of f and f', when plotted on a speed base, are as sketched. At any speed

$$\frac{m_w}{m_s} = \frac{ab}{bc}.$$

In a perfect liquid this would be constant at all speeds; but on account of the method in which f' is deduced, and on account of the ship running into a different tide to the log—there being no check by the *Active's* screw—such constancy could hardly be expected. The average of this ratio must therefore be taken.

In any particular experiment the curve of f' was somewhat irregular, and the curve (Fig. 59) is the mean of four curves at

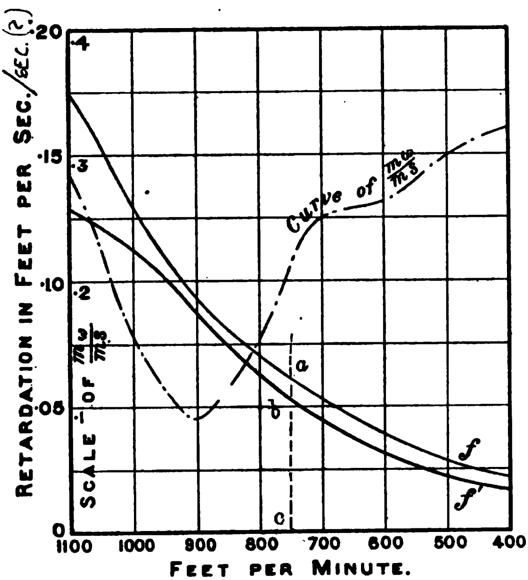

FIG. 59.

normal displacement but different times. The curve of f is taken from the same figure; and, knowing the mass of the ship, the resistance could be deduced.

It will be seen that the results vary considerably. The mean

of all the results is 0·225. Mr. W. Froude gave 0·2 for deep draughts, and 0·16 for higher draughts.

Acceleration experiments were not quite satisfactory, on account of the alternate slackening and tightening of the rope. They gave results about one-third the above.

§ 61. **Results of Trials.**—The ship was tested under different draughts, and under different trims for each draught. The particulars were—

Draught	13′ 9″	13′ 0″	12′ 1″
Surface in square feet	7540	7260	6940
Displacement in tons	1160	1050	938
Length	172′ 6″		
Beam	33′ 2″		
Trims	Varied from 1′ 6″ by the head to 4′ 6″ by the stern		

The speeds varied from 3 to 12½ knots. The normal displacement was 7540 tons, and the normal trim was 13 feet 6 inches forward and 14 feet aft. Under these conditions the ship was tried with and without bilge keels, the keels being two in number, each 100 feet long and 3 feet 6 inches wide. The change of trim for each displacement did not alter the wetted surface. The entrance and runs were finer the less the draught.

The principal objects of the experiments were to test the law of comparison, and to find the effect on the resistance of speed, trim, immersion, and bilge keels.

(1) *Effect of speed on resistance.*—Up to 8 knots, $R \propto V^2$ almost exactly, and with normal displacement and trim could be expressed by $88V^2$. Above 8 knots, it increases at a greater rate, so that at about 12·8 knots (1280 feet per minute), instead of being 14,400 pounds, as would have been the case if the law of square held, it

was 24,000 pounds. The tow-rope strain at different speeds was—

Speed in feet per minute	400	560	720	880	1040	1200	1230
Tow-rope strain in pounds	1001	1940	3420	5900	10100	17400	23300

(2) *Effect of alteration in trim.*—Alteration of trim did not appear to affect the resistance to any great extent. Between 8 to 10 knots there was no appreciable difference under extreme trims. Generally, as the ship is down by the head, the resistance appears to be increased at the higher and diminished at the lower speeds. Comparing the resistances under the extreme conditions of the ship ("by the head" and "greatly by the stern"), the difference at 12 knots was from 7 to 8 per cent., and at 4 knots about 10 per cent. in the opposite direction. With bilge keels added, however, the advantage of the ship at high speeds when trimmed "greatly by the stern" is not sustained.

(3) *Effect of immersion.*—The resistance was decidedly less at light than at normal draught. With light draught, between speeds of 8 to 12 knots, there is 10½ per cent. less resistance than at deep draught, the reduction in wetted surface being about 8 per cent. and in displacement 19¼ per cent. The reduction in displacement is therefore very much greater than the reduction in resistance. This increased displacement does not cause the total resistance to increase at anything like the same rate, and the total resistance increases at only a slightly greater rate than skin friction. This would appear to show that the wave making increased, but only slightly with draught—a result which might have been anticipated seeing that wave making is a surface effect—thus pointing out the economy of deep-draught vessels.

(4) *Bilge keels.*—The extra resistance when these were fitted was less than that caused by the skin friction alone. At 10 knots the extra resistance thus caused would have been 800 pounds, whereas the average increased resistance between 8 and 12 knots was only 350 pounds. This would appear to indicate that the

surface changed slightly for the better. If this may be taken as correct, the mean coefficient of friction is reduced.[1]

A second reason why the coefficient of friction is reduced is that a ribbed surface may cause a greater surface wake and therefore a reduction in the value of f; but, according to Mr. Froude, it seemed impossible to make the new value from 0·8 to 0·9 of the old for the *whole surface*. This fraction would get less, but only slightly less, the higher the speed.

§ 62. **Test of the Law of Comparison.**—The curves for the *Greyhound* have been discussed in § 58. The curve AA represents the total resistance of the model, the curve BB represents the wave-making resistance of the model. On a different scale, BB represents the wave-making resistance of the ship, and CC represents the total resistance of the ship. The curve DD represents the actual curve of resistance. It will be noticed that it lies above the curve deduced from model experiments, and that apparently the law of comparison is not strictly verified. It must be remembered that the coefficient of friction has been assumed that of a varnished surface. The *Greyhound's* bottom was of copper, which was deteriorated by age, and for which therefore a higher coefficient ought to be taken. The test, therefore, of the law of comparison will depend upon whether, keeping the wave making as deduced from the model unaltered, the ratio of the vertical distances between the curves D and B, and C and B, in Fig. 57, is a constant ratio.

The following table gives the ratio at different speeds :—

Speed in feet per minute	700	800	900	1000	1100	1200
Ratio	1·36	1·39	1·36	1·30	1·33	1·30

Thus the ratio is constant, and is equal to 1·33 times that of a varnished surface.

[1] Thus, at 10 knots, $f = 0{\cdot}009$; R_f, without keels = 4545; with keels, with same coefficient, 843 pounds, making a total of 5380 pounds. The increase resistance is only 350 pounds, making an actual total of 4890 pounds. Thus the new average coefficient is $\frac{4890}{5383} \times 0{\cdot}008 = 0{\cdot}00816$.

CHAPTER IV

WAVE-MAKING RESISTANCE

§ 63. **General Considerations.**—Wave-making resistance is the nett fore-and-aft resultant of the fluid pressures acting normally on all parts of the surface of the vessel. If a body is at rest in undisturbed fluid, the pressure at every point is the hydrostatic pressure, and the nett fore-and-aft effect is zero. If the body is travelling through the fluid, but deep below the surface, the pressures are largely changed from the hydrostatic pressures, in virtue of stream line action, but still the nett fore-and-aft effect is zero. If the body is travelling at or close to the surface, the pressures are still further changed from the hydrostatic pressures in virtue of the formation of waves, and such additional difference as is thereby introduced between the sum of the fore and aft pressures is the wave-making resistance. The approach to the surface of the fluid, by admitting of wave formation, has changed the pressures, because the wave system is really a changed set of stream lines, and involves a correspondingly changed set of pressures, the disposition of streams and pressures being throughout such that there is a perfect correspondence between the force acting on every particle, and the motion thereby impressed upon it. It does not by any means follow that the change in the pressures is throughout in the direction of increase of resistance, *i.e.* of increase in the sternward pressures, and decrease in the forward pressures, for probably in most cases the change of pressure is of the nature of a large forward force on some parts of the surface, and a larger sternward force on others, the wave-making resistance being the difference

between these two. But it is, of course, an essential condition that this nett sternward effect of the change of pressures should be such that the increase of energy consumed in propelling the body against the increase of resistance, should equal the energy demanded by the maintenance of the wave system.

The ship thus tends to generate two principal wave systems—one forward, and the other aft. Each primary wave is followed by a train of waves of gradually diminishing heights. In addition, there might be minor systems, as, for example, due to diminution of pressure amidships. With a long middle body, the diminution of pressure takes place so slowly, and is distributed over such a long length, as to make the wave system due to it inappreciable; but the greater the rate of diminution of pressure, the more important do these minor systems become. The same remark applies to the bow and stern waves. The forms of the entrance and run have an immediate effect on the heights of the waves formed, and therefore on the wave-making resistance.

§ 64. **Wave Patterns.**—The main characteristics of the wave systems formed seem to be the same under all conditions[1] (Fig. 60). It invariably consists partly of transverse, and partly of diverging waves. Both systems owe their origin to the general excess pressure at the bow and stern. With a ship going at a definite speed, a definite wave system is produced. To a person on board the ship, the system surrounding the ship appears to travel with the ship. Consequently, the length of the transverse waves from crest to crest is that corresponding to the length of a deep-water wave travelling at the speed of the ship (§ 28); and the distance between the diverging waves, normal to the crests, that corresponding to a speed equal to that of the ship's speed in that direction.

The bow transverse waves which are square, or nearly so, to the line of advance, show for a considerable distance along the side of the ship, the heights diminishing as they speed sideways. If the middle body be very long, the bow transverse series will disappear—due to spreading—before the stern is reached, and the stern will then leave a series of transverse waves of its own, of a character

[1] Mr. W. Froude, *Transactions of the Institution of Naval Architects*, 1877.

similar to those formed at the bow; if, however, the ship be not long enough, the two series will coalesce into one. Hence, it will be noticed that the bow waves do not become dissociated, immediately after their formation, from the ship, and consequently they will still probably affect the resistance even after being formed.

As will be seen from Fig. 60, the diverging waves consist of a number of short ridges stepped back so to form a row "en echelon." They taper each way from the middle towards the ends, and, so far as can be judged, their wave length is that appropriate to a wave travelling at a speed equal to that component of the ship's speed

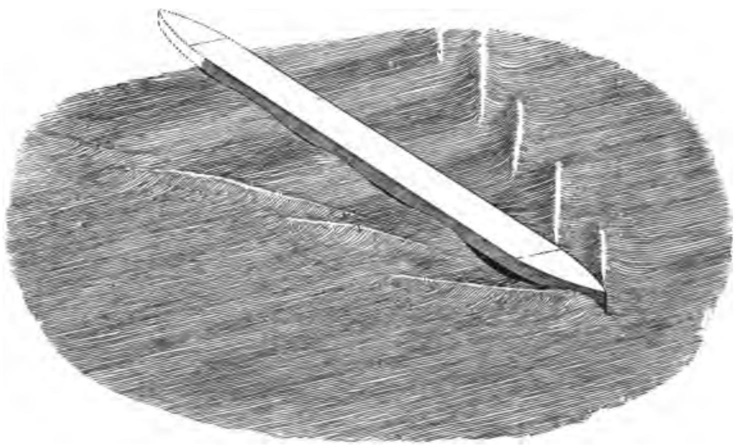

FIG. 60.

taken normally to the crest line. The general line of the series, that is, a line drawn in plan through the highest points of successive individual crests, diverges from the line of motion at an angle large enough to place clear of the ship's side all the diverging waves formed by the bow except the first, and sometimes the very innermost ends of the second. The diverging waves become, therefore, dissociated from the ship, and, when once produced, probably produce no further effect on the resistance.

The principal diverging series is formed at the bow, but a series precisely similar in character, though generally less marked, is produced by the stern. With the same vessel, run at different

speeds, the angle of divergence of the diverging waves appears to be sensibly constant. Hence they increase in length relatively to the ship, succeeding waves dropping further and further astern as the speed increases.

The stern diverging waves likewise drop further astern, and the primary wave of the system becomes more acute with rise of speed. For a similar reason, the transverse waves increase in length.

Further, experiment shows that it can only be done so by some rearrangement of stream line pressures. In other words, if anything happens to the bow series due to the after body, the stern wave system must be affected. The bow wave in being absorbed must affect the stern system, and, probably, it averts its formation. The residue of the bow system interferes with what would be the stern series, and it is this interference which is the cause of the humps and hollows.

§ 65. **Illustrations of Wave Patterns.**—The angle of divergence of the diverging waves, and the relative importance of the trans-

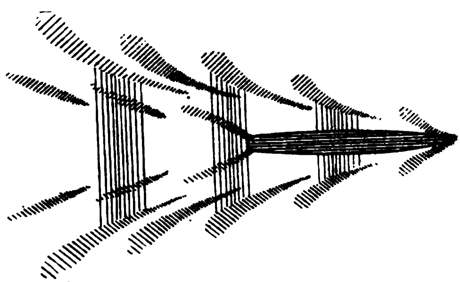

Fig. 61.

verse and diverging series, vary of course in different ships, and in the same ships at different speeds; the general characteristics of the systems, however, as above described, seem common to all forms of vessels under all circumstances. It is instructive to trace, step by step, the train of modifications whereby the wave system accompanying a large ship at ordinary full speed, or its equivalent, that accompanying a torpedo launch at low speed, changes into the system accompanying the same launch at her full speed. With this object a careful survey was made of the plan of the wave

system accompanying a model of a torpedo-boat recently tried at Torquay at a large range of speeds. At certain speeds, also, the longitudinal section of the level of water surface in the wake was

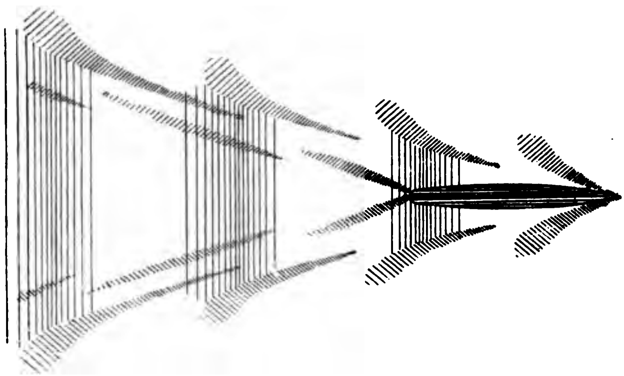

FIG. 62.

observed, by measuring downwards from a carriage following the model on the level railway of the experimental tank. The wave

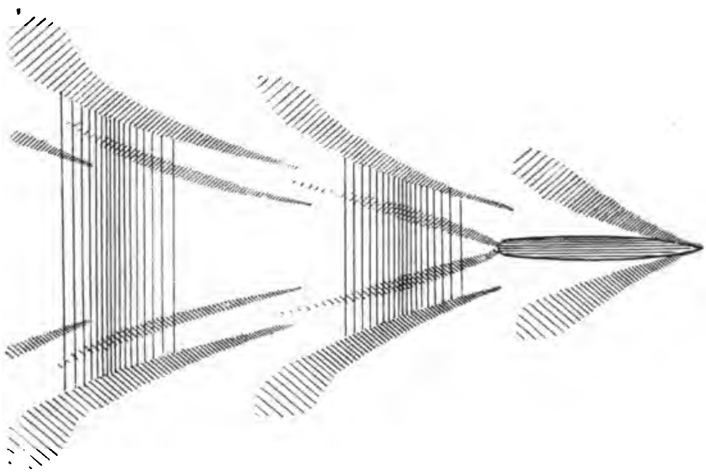

FIG. 63.

patterns of an 83-feet launch at five speeds are shown in Figs. 61–65, at speeds of 9, 12, 15, 18, and 21 knots, the position of the wave crests being indicated by shading. Figs. 66, 67 show in

comparison, on the same scale, the system made at a speed of 18 knots by the 83-feet launch, and a boat of the same lines, 333 feet long.

It will be seen that at the 9-knots speed for the 83-feet launch

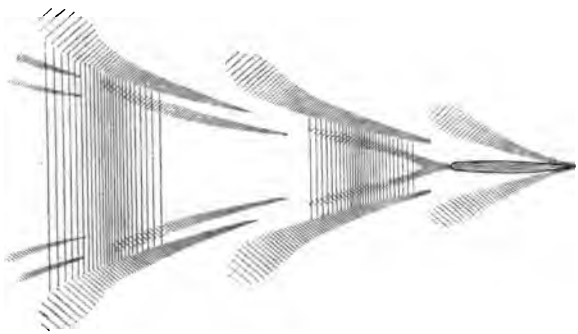

Fig. 64.

(Fig. 61), or the 18-knots speed for the 333-feet ship (Fig. 67), the wave system is precisely of the character observable on large ships, at full speed, showing the familiar train of diverging waves at the bow and at the stern. As the speed is increased, or the size decreased,

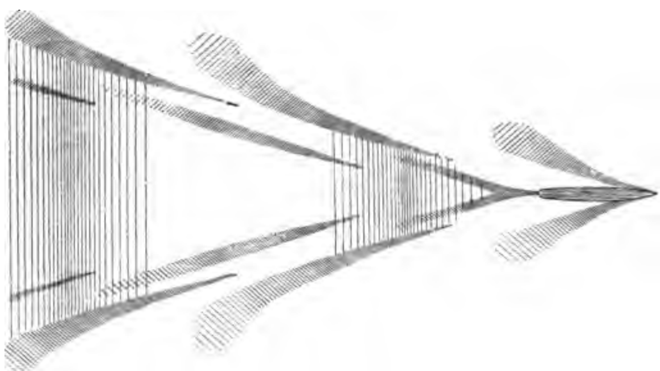

Fig. 65.

both trains of diverging waves retain their character, but expand in scale relatively to the size of the ship, as they necessarily must—since the angle remains practically the same—in order that their length may suit the speed; so that at 12 knots for the 83-feet

boat the second diverging wave (that is, the first echo of the wave at the bow) is nearly opposite the stern; at 15 knots, more than

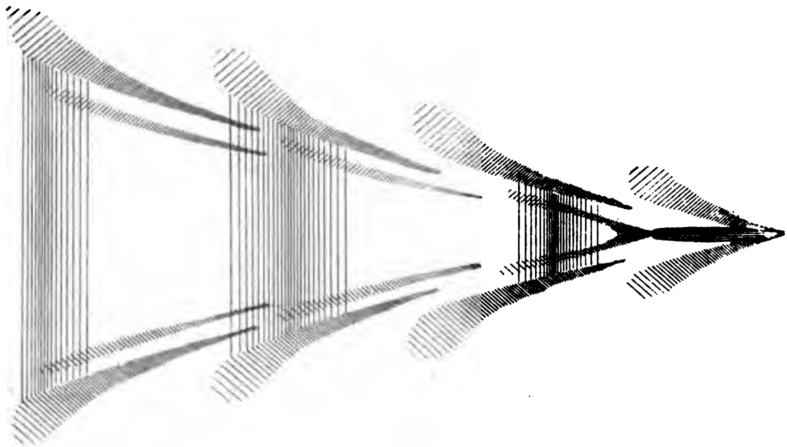

FIG. 66.

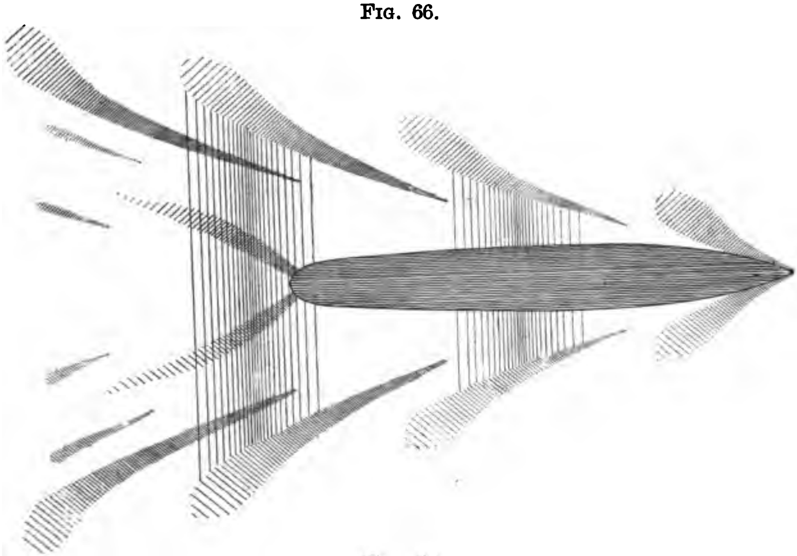

FIG. 67.

half a length clear of the stern; and at 21 knots, nearly two lengths clear. The point of departure of the first stern diverging wave drops astern as the speed increases, and it becomes more acute at

its forward end; at the higher speeds it is recognizable as the peculiar kind of flat-topped cliff of water, wedge-shaped in places, which always appears immediately astern of a high-speed launch, and which is now seen to be the representative of the first member of the ordinary stern diverging wave series.

The transverse waves left in the wake were very low and flat at high speeds, and were invisible to the eye—in the model—above the 15-knots speed for the 83-feet boat, but they show plainly in the longitudinal section of the wake at the 18 and $20\frac{1}{4}$ knots speeds, and are found to have the correct length appropriate to the speed.

§ **66. Experimental Work—Effect of adding Middle Body as the Total Resistance.**—Mr W. Froude[1] made experiments representing a series of imaginary ships of identical cross-section and identical form of ends, the difference between them consisting in the length of parallel body, of uniform cross-section, inserted amidships. The dimensions of the largest ship are: Beam, 38·4 feet; draught, 14·4 feet; length of fore body, 80 feet; length of after body, 80 feet; parallel middle body, 340 feet; total length, 500 feet. The dimension of the shortest ship is 160 feet, so that it has no middle body. The experiments were very searching, as observations were made on lengths of middle body decreasing by 20 feet, and for below 60 feet by 10 feet. In fact, each curve was plotted by twenty points. This was necessary in order to obtain the curves of resistance to the greatest accuracy.

The models of the ships were all made by actually shortening them amidships (cutting out the necessary length of middle body), and rejoining the ends. This was done partly for convenience and partly to ensure identity in frictional quality of skin between the different members of the series.

The results were plotted as a series of curves, the ordinates being total resistance in tons, and the abscissæ speed in knots. They are shown in Fig. 68.

The ships whose curves are shown range from 160 feet to 480 feet in total length, and consequently from 0 to 320 feet in length of straight, by intervals of 40 feet. Their displacements range

[1] *Transactions of the Institution of Naval Architects*, 1877.

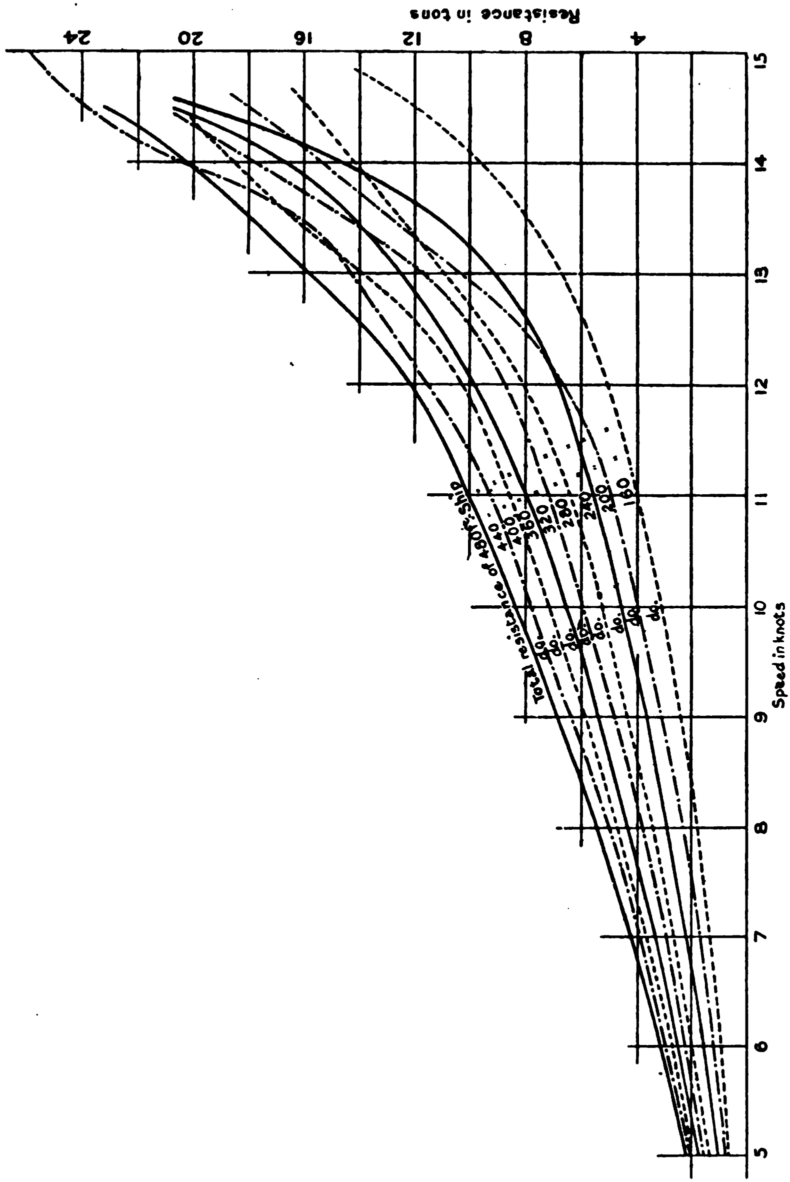
Resistance in tons
24
20
16
12
8
4
Speed in knots
5
6
7
8
9
10
11
12
13
14
15
Total resistance of 480 ft. Ship
440 do.
400 do.
360 do.
320 do.
280 do.
240 do.
200 do.
160 do.

FIG. 68.

from 1245 tons to 5938 tons, by intervals of about 142 tons.

Comparing together the curves of resistance of these ships, it will be noticed that at the lower speeds every added 40 feet of length (and 568·7 tons of displacement) increases the resistance by the same amount; but at higher speeds this harmony disappears. At 13 knots, for example, the 200-feet ship makes considerably more resistance than the 240-feet ship, which has 568 tons more displacement; and, though at 14½ knots the larger ship, again, makes the greater resistance, yet, even at 14 knots, the 280-feet ship makes less resistance also than the 200-feet ship of 1137 tons less displacement, and the 240-feet ship of 568 tons less displacement; and at 14½ knots the 200-feet ship makes almost as much resistance as the 360-feet ship, of 2275 tons more displacement. Similar anomalies appear in the comparison between other ships. The tendency to alternate excesses and defects of resistance in the shorter ships, as compared with the larger, appears throughout the diagram.

§ 67. Curves of Residuary Resistance.—Now, regarding the resistance of a ship as made up in three items, namely, skin friction, eddy-making resistance, and wave-making resistance; and since the first resistance is approximately proportional to the area of the skin, so that additions of successive equal increments of parallel side can only affect it to the extent of producing corresponding equal increments for every additional length,—the anomalies can only be due to some unexpected effect which the distance between the two ends produces upon the other two items which make up what may be conveniently termed the "residuary resistance." To analyze the nature of this effect the curves of residuary resistance must be drawn.

In Fig. 69 the results of all the series of ships are represented by curves, the ordinates above the zero line AA represent the "residuary resistance." The abscissæ represent the length of parallel side. Thus the ordinates to the spots on the vertical line BB are the "residuary resistances" of the 160-feet ship having no parallel side, at several speeds, 6·75, 9·31, 11·31, 12·51, 13·15, 13·79 and 14·43 knots. The series of spots next to

the left indicate the "residuary" resistance at the same speeds of the 170-feet ship, having 10 feet parallel side; and so on, the distances to the left of the zero line BB being length of parallel side on the scale of 80 feet to an inch.

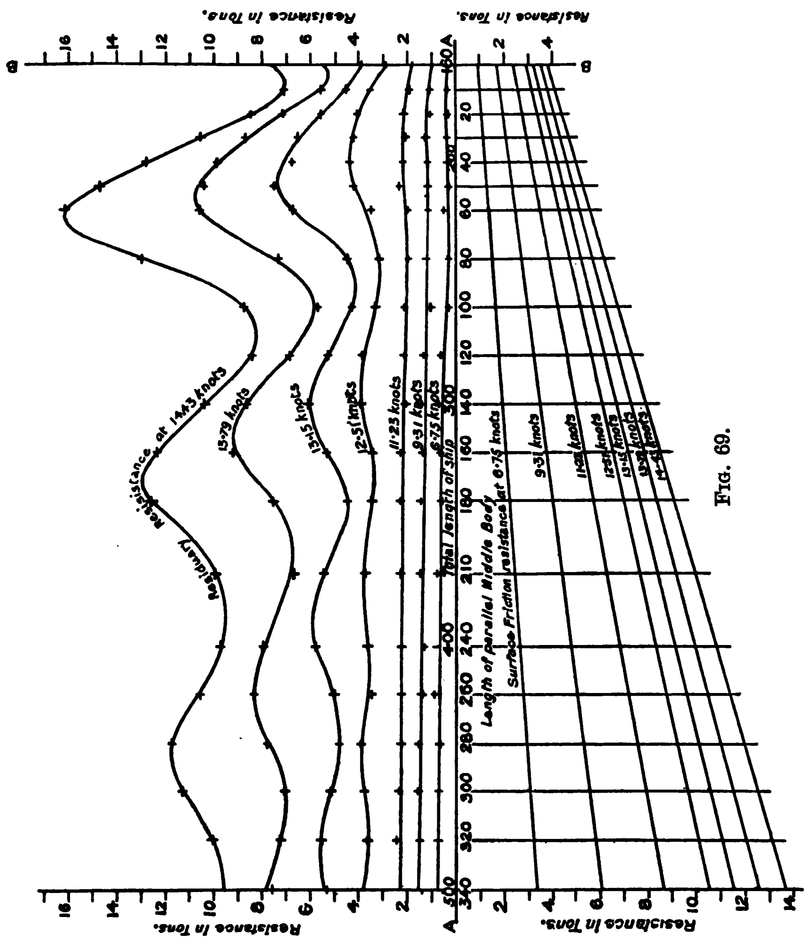

FIG. 69.

Through the series of spots representing the "residuary resistances" of the series of models at each speed curved lines are drawn, each of which represents the gradual change in "residuary

resistance" corresponding to gradual elongation at a particular speed, the ordinates to any one of the curves at any intermediate point between the spots being the probable "residuary resistance" of the ship having length of parallel side represented by the corresponding abscissa, at the speed belonging to the curve.

In the same manner the curves below the horizontal zero line AA represent the change in the surface friction element due to elongation; the ordinates to these curves (measured downwards from the zero line AA) being, at the stated speeds, the skin friction resistance of the ships having length of parallel side corresponding to the abscissæ; so that, measuring the total ordinate, from any of the spots representing "residuary resistance" of a certain ship at a certain speed, down to the surface friction curve for the same speed, will give the total resistance of that ship, and so supply, if necessary, the information omitted from Fig. 68.

Considering only the "residuary resistance," the undulations present the following characteristics:—

(1) The spacing, or length of the undulation, appears uniform throughout each curve.

(2) The space is more open in the curves of the higher speed, the lengths being apparently about proportional to the square of the speed.

(3) The amplitudes, or heights of the undulations, are greater in the curves of higher speed.

(4) The amplitude, in each curve, diminishes as the length of parallel side increases.

§ 68. **Compound Wave Systems.**—Theoretically, with a given entrance and run, certain waves are formed, and it is very desirable to discover the reasons why the length of the middle body and speed affect the residuary resistance. To account for these fluctuations—which, for a given speed, are more pronounced the shorter the middle body; and, for a given length, are more pronounced the greater the speed—Mr. W. Froude assumed that they were due to the position of the bow wave series relative to the run. If a crest coincided with the run, he inferred there was a favouring effect; whilst if a hollow coincided with the run, it caused a

retarding effect, the action of the after body to make waves on its own account going on meanwhile unimpeded.

Mr. R. E. Froude found that his experiments were not satisfied by the theory stated. He pointed out that it was virtually a case of interference. It is clear that the stern waves must be affected by the residue of the bow waves, for the echoes of the primary bow wave modify the pressures along the modified stream lines, and must, therefore, modify the formation of the stern series. The humps and hollows show that at some speeds, some energy is reabsorbed, and at others not reabsorbed, or, may be, reversed. Both bow and stern waves are formed on account of stream line action; and, if either one or other is absorbed, it can only be done so by some rearrangement of stream line pressure. The bow wave in being absorbed must do something to the other, and it suggests itself, as a reasonable hypothesis, that this action is to arrest its formation. The residue of the bow wave system interferes with what would be the stern series, and it is this interference which is the cause of the humps and hollows.

§ 69. **Mechanical Illustration of Interference.**—Mr. R. E. Froude suggests a mechanical explanation of interference.

Imagine a pendulum or plumb-bob fastened to a ring travelling along a frictional rod at a uniform speed. Let the rod be bent transversely in two places by S-curves, as at AA, BB (Fig. 70), the two straight parts at each end being in the same straight line, and the middle straight part AB parallel to them. When the ring travels on the part AABB, it will be first displaced sideways in one direction, will remain in this new position for a certain time, and be eventually replaced in its original position. The first displacement will get up a lateral swing in the pendulum—greater or less, according as the relations between the natural period of the swing of the pendulum and the time occupied in the displacement —and this swing will continue unaltered as long as the ring remains on the middle straight part. This swing represents the transverse wave series left by the bow, which shows unaltered all along the parallel side, except so far as it diminishes by spreading sideways. If the pendulum be artificially stilled before the second curve is reached, the replacement will likewise generate a swing

which will remain unaltered throughout the succeeding straight part, and represent the train of independent transverse waves left by the stern in a vessel with very long parallel sides.

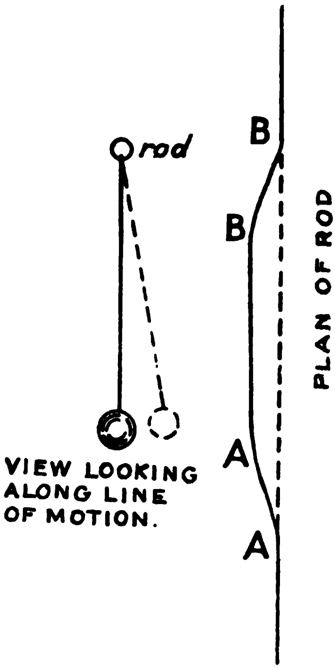

Fig. 70.

If, however, the pendulum remains swinging when the second curve arrives, the behaviour of the pendulum on it, and the magnitude of the resulting swing, will depend entirely upon the point in its vibration which it has reached at the moment of commencing the second curve. The solution of the problem is somewhat complex. In one simple case, it is very simple. If the two curves AA, BB are exactly symmetrical with one another, and if the length of the middle straight is so chosen that the pendulum enters the second curve in the attitude and state of motion symmetrical to that in which it left the first, then the behaviour of the pendulum throughout the journey over the second curve will be likewise symmetrical to its behaviour on the first curve, and it must, therefore, leave the former as it entered the latter, namely, in a state of rest.

Generally, the resulting swing is a compound of two others, namely, that which would have remained if the second curve had not existed, and that which the second curve would have created if the previous swing had not existed; and these component swings being simple harmonics of the same period as the two components, it follows that when the two components are simultaneous in the same direction, the resulting vibration will be at its largest, and will be the sum of the two, the energy being proportional to the square of that swing; when the two components are opposite to one another the resultant will be at its smallest, being equal to their difference.

§ 70. **Analysis of Resultant Wave System.**—In dealing with transverse waves, the actual system astern of the boat is the resultant of two component systems, one of them imaginary, namely, the system which remains of the bow wave system when the stern is reached, and the stern wave system which would be formed had the bow wave system disappeared by the time the stern is reached. The natural stern wave system is not actually formed, but, as a net effect, the resulting system may be considered the resultant of the above two components. The after body, has, in fact, a two-fold effect, namely, it causes the absorption of the bow wave system, or of what remains of it; and, in the process of absorption, causes the complete or partial frustration of the natural stern wave.

§ 71. **Principal Effects of Interference.**—For simplicity, it may be assumed that the natural bow and stern waves are simple waves of heights h_1, h_2. Let the height of the bow wave, when it has passed aft, be kh_1, in which k is a number less than unity. Let s be the distance (Fig. 71), measured along the ship, between what would be the primary crests of the natural bow and stern systems, assuming that each was formed independently of the other. Moreover, let s' be the distance between the primary natural stern waves and the nearest crest of the bow series; and let λ be the wave length, which is the same for each series. Thus the resulting system at the stern must have the same length as the component systems, and (§ 42) the height h is given by

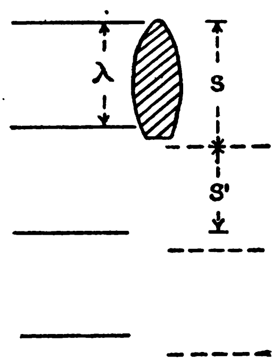

FIG. 71.

$$h^2 = k^2h_1^2 + h_2^2 + 2kh_1h_2 \cos \frac{2\pi s'}{\lambda}.$$

Since $(s + s')$ is some multiple of λ, this may be written

$$h^2 = k^2h_1^2 + h_2^2 + 2kh_1h_2 \cos \frac{2\pi s}{\lambda}.$$

Let L be the length of the boat, and let $s = m$L, in which m

will not differ greatly from unity, and which might slightly increase with speed. Then, if v be the velocity of the ship in feet per second

$$\lambda = \frac{2\pi v^2}{g} \quad (\S\ 29)$$

and

$$h^2 = k_1^2 h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{gmL}{v^2}.$$

The rate at which energy drains away from the ship has been discussed in §§ 39, 40. The energy of a wave, per foot breadth, is proportional to the length and as the square of the height. At the bow the energy, per foot breadth, is proportional to λh_1^2, and aft of the bow system is proportional to $\lambda k^2 h_1^2$, the energy lost being proportional to $\lambda(h_1^2 - k^2h_1^2)$. The energy of the combined system is proportional to λh^2, so that the total energy per wave length, per foot breadth lost, is proportional to

$$\lambda\left(h_1^2 - k^2h_1^2 + k^2h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{gmL}{v^2}\right)$$
$$= \lambda\left(h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{gmL}{v^2}\right).$$

The resistance per foot breadth will be proportional to this, but divided by λ, so that resistance per foot breadth, due to the fluctuating term, is proportional to

$$h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{gmL}{v^2}.$$

§ **72. Discussion of Result.**—With a given ship run at gradually increasing speeds, L is constant, m might possibly slightly increase with speed, but may be assumed constant and equal to unity. The heights h_1, h_2 and the coefficient k will increase with speed; for at low speeds k will be practically zero, and at very high speeds sensibly equal to unity. The expression

$$\frac{gmL}{v^2}$$

will continually decrease, and the term

$$2kh_1h_2 \cos\frac{gmL}{v^2}$$

will consequently fluctuate. Thus, then, for a given ship run at increasing speeds, the resistance fluctuates about an ever-increasing mean value, the magnitudes of the fluctuations depending on kh_1h_2, and therefore increasing; and the spacings of the fluctuations being given by decrements of $\frac{gm\mathrm{L}}{v^2}$ of 2π. Since m and L are constants, the speeds corresponding to successive humps or hollows are such that the squares of the speeds are in harmonical progression. Thus in one experiment the humps occurred at 34 and 21·5 knots approximate. The next hump will occur at

$$\frac{1}{v^2} = \frac{2}{21{\cdot}5^2} - \frac{1}{34^2} = \frac{1}{284}$$

that is, at 16·8 knots. This was the observed speed. It will be noticed that the interval between 34 and 21·5 is much greater than the interval between 21 and 16·8, thus verifying the experimental statement in § 67.

Again, with a given entrance and run, and at a given speed, h_1, h_2, and m will be constant, but k will decrease as the length of the middle body is increased. Thus, in such a case, the resistance will not be constant, but will oscillate about a mean value, the amplitudes of the oscillations decreasing as the length of the middle body, depending, as it does, on k; and the spacings such that the increments of $\frac{gm\mathrm{L}}{v^2}$ differ by 2π, so that the spacings at a given speed in a length base will be uniform, and at different speeds will vary as v^2. Those verify the experimental observation of Mr. W. Froude (§ 67). Thus at 14·43 knots the calculated spacing is 116 feet—actually, it was 110 feet; or at 13·15 knots is 96 feet—actually, it was 92 feet.

Finally, in similar vessels run at corresponding speeds, $\frac{v^2}{\mathrm{L}}$ is constant and may be written c^2, so that the wave-making resistance per foot breadth may be written

$$h_1^2 + h_2^2 + 2kh_1h_2 \cos\frac{gm}{c^2}.$$

Since, at corresponding speeds, the wave patterns are merely

reproduced on different scales, k and m must be the same, so that the fluctuations will exactly correspond at corresponding speeds.

In vessels of different types—such as the mercantile marine, battleships, destroyers, etc.—the full speed values of c will probably be tolerably constant for each type, but will vary with different types, being least for the mercantile marine, and greatest for destroyers. The spacing of the fluctuations will depend on the 2π increment of $\frac{gm}{c^2}$, and therefore for different values of c will increase as c increases. In other words, for small values of c, a slight change in value might run from a hump to a hollow, whilst for large values this would not be the case. In the mercantile marine, where c has a comparatively small value, the coefficient kh_1h_2 is so small that there is no material alteration in the resistance, whether it be on a hump or a hollow; and for destroyers, although the coefficient kh_1h_2 is large, yet a considerable alteration in c does not materially alter the position on the curve, and so, again, the resistance may be approximately calculated. But in the intermediate case, as, for example, in cruisers and battleships, the coefficient is tolerably large, and the fluctuation is tolerably rapid. It is, therefore, difficult to express the wave-making resistance in such a case.

§ 73. **Analysis of the Diverging Systems.**—The formation of the "echelon" waves is a further illustration of a combined wave system, but is more complex than the simple transverse systems. In § 71, two transverse wave systems were discussed; the first, when two systems of the same length, and therefore of the same velocity, of different phases, coalesced. This combination was used in discussing the theory of interference on the "humps and hollows" in the curve of residuary resistance.

So in the diverging series. The diverging waves (§ 64) practically leave the ship, and do not affect the resistance once they are formed; and practically the bow and stern series are independent, each forming wave systems of their own. But these represent a considerable amount of energy, which has to be added to the wave-making resistance of the transverse system, and also to the skin resistance.

We also discussed at length the question of group velocity (§ 42), and found that in deep water the rate at which energy was transmitted was half the velocity of the group. One method was to calculate the energy of the system (§ 41); the other method was to combine two wave systems of slightly different wave lengths, the direction and velocities of propagation being so related in each case that there is no change of position relatively to the boat.

This method may be applied to the diverging system. Let (Fig. 73) A, A . . ., B, B . . . be the respective crest lines pro-

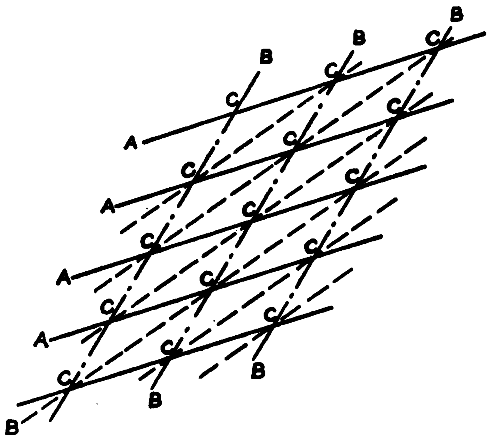

Fig. 72.

ceeding towards the stern, the waves all travelling in the same direction and with the same velocity, but inclined to the ship. The primary wave produces "echoes," and these coalesce with the other diverging waves. In Fig. 73 the positions of the crests are marked by C, C, C . . . Along A the waves will be steeper than along B. The locus of maximum crests is the line *ccc*, three of which are shown (Fig. 72).

Imagine the wave crests A and B are brought, by degrees, more parallel to each other and equal in length. The general characteristics will remain unaltered, but the length of the waves along A and B will be much increased. Fig. 73 shows the effect

of making the angle small. In the limit, when the wave crests become sensibly parallel, the hollows are at an infinite distance on either side, and the broken line C represents the line of

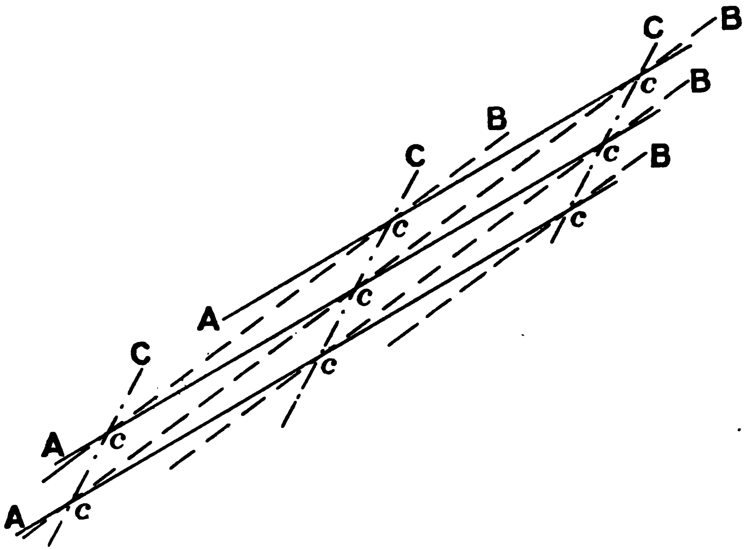

FIG. 73.

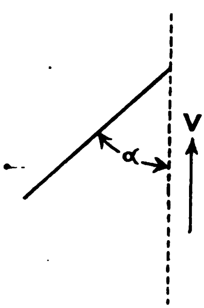

FIG. 74.

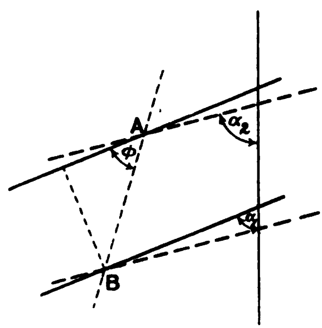

FIG. 75.

maximum crests. And these points are stepped back, one behind the other, the wave length of the resultant system being the same as either component.

Let (Fig. 74) the locus of the crest of the "echelon" wave be

inclined at an angle a to the direction of the ship's motion, and let V be the speed of the boat. The speed of the diverging wave is

$$V \sin a$$

and therefore their length is

$$\frac{2\pi V^2 \sin^2 a}{g} = \lambda \sin^2 a \quad (\S\ 64)$$

where λ is the length of the transverse waves. In Fig. 75 the line of maximum crests is AB. Two crests, shown dotted, are taken passing through AB, and at a very small angle with the first line. Let a_1, a_2 be the angles which the full line crest and dotted line crests make with the line of motion, and denote $a_2 - a_1$ by θ, which is very small. The angle ϕ is the angle between the line of maximum crest with the crest of the wave. Clearly

$$\lambda \sin^2 a_2 = \text{AB} \sin (\phi + a_2 - a_1)$$

$$\lambda \sin^2 a_1 = \text{AB} \sin \phi$$

$$\therefore \frac{\sin^2 a_2}{\sin^2 a_1} = \frac{\sin (\phi + a_2 - a_1)}{\sin \phi}$$

or

$$\frac{\sin^2 (a_1 + \theta)}{\sin^2 a_1} = \frac{\sin (\phi + \theta)}{\sin \phi}$$

whence

$$1 + \frac{2 \sin a_1 \cos a_1 . \theta}{\sin^2 a_1} = 1 + \frac{\theta}{\tan \phi}$$

$$\therefore \tan \phi = \tfrac{1}{2} \tan a_1.$$

Thus the angle of the line of maximum crests is such that its tangent is half the tangent of the crest angles. Since both angles are fairly small, the angles may be written for the tangents.

§ **74. Vessels of Abnormal Form—Inclined Floats.**—The previous results refer to vessels of ordinary form. The possibility of obtaining high rates of velocity in ships by so shaping them that on being driven through the water they shall become more or less lifted, and thereby offer less resistance to motion by skimming along the surface, has frequently been pointed out. Occasionally it is suggested to "lift" the ship by machinery in the ship itself, which, besides propelling her, shall provide also a vertical reaction. This, however, is practically impossible.

This problem excited great attention in 1873, when Ramus persuaded the Admiralty to take the question up. The matter was referred to Mr. W. Froude, who made experiments on a model of Ramus's boat, and a boat of ordinary form and of the same displacement.

The form adopted consisted of two inclined planes, one abaft

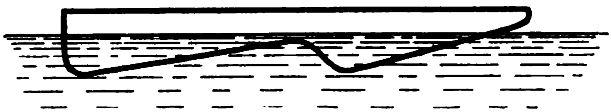

FIG. 76.

the other. When the second plane is immediately aft of the first, it works in the wake of the first, or in the water depression of the first. This causes the change of trim to be considerable, and, in order to obviate this difficulty, Mr. Froude arranged the floats as shown in Figs. 76, 77. Different-sized models were tried of this very peculiar-shaped boat, and the law of comparison was found

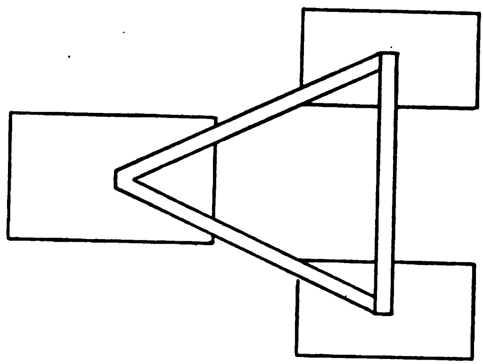

FIG. 77.

to be amply verified. The inventor claimed that with such a vessel very little increased resistance would be experienced with increased speed, and that the only limit of speed would be due to atmospheric resistance.

§ 75. **Mechanical Principles.**—Mr. W. Froude considered the forces acting on a plane moving through the water at a small angle to the direction of motion. When the vessel floats at rest in water,

the weight of the vessel is exactly equal to the weight of water displaced. When it moves on the surface of water, the pressures are quite different to the static pressures, and the displacement need not be the same as when at rest, and when the vessel moves at a sufficiently high speed, it was assumed that the dynamic pressures thus introduced would support the vessel on the surface of the water, so that, practically, no water was displaced. Now, suppose that this state of affairs can be obtained, that is, the vessel skims along the water, being supported by the dynamic pressures, and, further, suppose there is no wave-making resistance. Consider the forces acting (Fig. 78); as the plane BC moves forward, the forces operating are the normal pressure P and the tangential force F. The resistance to motion consists of two parts—

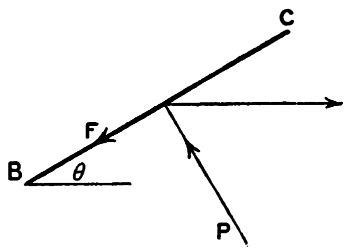

FIG. 78.

(1) The horizontal component of the normal pressure on the plane; and

(2) that of the surface friction of so much of the surface as remains immersed.

The vertical component of these forces must be equal to the weight.

Let (Fig. 78) S = wetted surface in square feet;
V = forward speed in knots;
V′ = velocity of gliding over CB, in knots;
P = normal force;
F = frictional force along plane.

Then for small angles of θ,

$$\text{normal pressure} = a\text{AV}^2 \sin \theta$$

where, for a totally immersed body, $a = 1{\cdot}7$ when the speed is in feet per second, or 4·8 when the speed is in knots, so that a may be taken 3 instead of 4·8.

Thus, then,

$$\text{P} = 3\text{AV}^2 \sin \theta.$$

Also
$$\begin{aligned} F &= fAV_1^2 \\ &= 0{\cdot}01AV_1^2, \text{ practically} \\ &= 0{\cdot}01AV^2 \cos^2\theta \\ &= \frac{P}{300}\cdot\frac{\cos^2\theta}{\sin\theta}, \text{ by substitution.} \end{aligned}$$

Hence, if W = weight,
$$\begin{aligned} W &= P\cos\theta - F\sin\theta \\ &= P\left(\cos\theta - \frac{\cos^2\theta}{300}\right) \end{aligned}$$

or
$$P = \frac{W}{\cos\theta\left(1 - \dfrac{\cos\theta}{300}\right)} \quad \text{and } F = \frac{W}{300}\cdot\frac{\cot\theta}{1 - \dfrac{\cos\theta}{300}}$$

giving P and F in terms of W and θ.

The resistance R, in pounds, is
$$\begin{aligned} R &= P\sin\theta + F\cos\theta \\ &= W\left\{\frac{\tan\theta}{1 - \dfrac{\cos\theta}{300}} + \frac{1}{300}\cdot\frac{\cot\theta\cos\theta}{1 - \dfrac{1}{300}\cos\theta}\right\} \end{aligned}$$

Since θ is small, this may be written
$$R = W\left(\theta + \frac{1}{300\,\theta}\right).$$

Thus R is a minimum when $\theta^2 = \dfrac{1}{300}$ or $\theta = \dfrac{1}{17{\cdot}3}$, and is then $\dfrac{1}{8{\cdot}6}$ of the displacement.

Thus, when the critical state is reached, neglecting wave-making resistance, the minimum resistance is $\dfrac{1}{8{\cdot}6}$ of the displacement, and is constant at all speeds. It will be shown in the next chapter that this is greater than that in high-speed boats, such as destroyers. Thus the idea of obliterating water resistance by the use of inclined planes is obviously false, even when the angle of the plane is indefinitely reduced.

§ 76. **Experimental Results.**—The experiments were made on a model of a ship of 2500 tons displacement, 360 feet long, 50 feet

beam, and 7 feet draught. The slope of the planes was 1 in 50, being that suggested by Ramus. Briefly, the results are:

(1) Up to 30 or 40 knots, there is no appreciable bodily lift, although the trim is altered—the bow being raised and the stern depressed.

(2) From this speed up to 130 knots, the bodily rise continually increases, being at the upper limit considerably more than half the displacement.

(3) There is no diminution of resistance within these limits, although, as the higher velocities were approached, the increase of resistance became much less rapid.

At the same time, with those abnormally formed models, a ship-shaped model of the same extreme dimensions, draught, and displacement was tried, and the results plotted in the same way. The results are as follows:—

Total horse-power (including air resistance) per ton.		
Knots.	Ramus's model.	Ordinary ship.
10	0·39	0·21
12	0·78	0·44
17·8	3·59	2·10
52	54·4	42·4
130	182·0	—

CHAPTER V

TRIALS ON FULL-SIZED SHIPS

HAVING discussed the general peculiarities of wave-making resistance, it is necessary to consider how the total resistance is affected by varying conditions. The most important problems may be summarized as follows :—

(1) The effect of increased size on the resistance, the vessels being similar in form.

(2) The effect of altering the form, the displacement remaining constant.

(3) The effect of altering the draught and displacement, the length and beam remaining unaltered; that is, the effect of loading.

(4) The effect of adding middle body and so increasing the displacement, the beam and draught remaining unaltered.

(5) The effect of the depth of water.

Any deductions which may be made must necessarily be comparative. The most important aspect of the question is from the point of view of economical propulsion. This may be best measured by stating the total resistance per ton of displacement, or the horse-power per ton of displacement, at varying speeds. It includes, therefore, the joint effect of skin friction and wave-making, and represents the resultant effect.

§ 77. **Laws of Comparison.**—The wave-making resistance of a ship will depend in some way on the velocity, and also upon the displacement. Let Δ represent the displacement, V the velocity, and R_w the wave-making resistance. The relationship may be expressed by the formula

$$R_w = a\Delta^m V^n$$

in which a, m, and n are coefficients determined from experiment.

For a smaller ship

$$r_w = a\delta^m v^n$$

whence

$$\frac{R_w}{r_w} = \left(\frac{\Delta}{\delta}\right)^m \left(\frac{V}{v}\right)^n.$$

The law of comparison gives

$$\frac{V}{v} = \sqrt{\frac{L}{l}} = \sqrt[6]{\frac{\Delta}{\delta}}$$

and

$$\frac{R_w}{r_w} = \frac{\Delta}{\delta}$$

so that, since the ships move in water of the same density,

$$\frac{\Delta}{\delta} = \left(\frac{\Delta}{\delta}\right)^m \left(\frac{\Delta}{\delta}\right)^{\frac{n}{6}}$$

or

$$\left(\frac{\Delta}{\delta}\right)^{m+\frac{n}{6}-1} = 1$$

that is,

$$m + \frac{n}{6} - 1 = 0.$$

Thus, if the resistance of a ship can be expressed by the formula, m and n are related by the equation given above. A table may be drawn up for different values of n.

n	0	1	2	3	4	5	6
m	1	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0
R_w	$a\Delta$	$a\Delta^{\frac{5}{6}}V$	$a\Delta^{\frac{2}{3}}V^2$	$a\Delta^{\frac{1}{2}}V^3$	$a\Delta^{\frac{1}{3}}V^4$	$a\Delta^{\frac{1}{6}}V^5$	aV^6

When $n = 0$, the wave-making resistance is proportinal to the displacement; when $n = 6$, it is independent of displacement.

The eddy-making resistance varies as the square of the velocity, and also the area of the cross-section (§ 49). It varies therefore as $\Delta^{\frac{2}{3}}V^2$, and follows the law of comparison. It may therefore be included with wave-making resistance.

The skin resistance (§ 51) follows the law

$$R_s = SV^{1 \cdot 83}$$
$$\propto \Delta^{\frac{2}{3}}V^{1 \cdot 83}$$

and cannot be included in the law of comparison. Moreover, the values of f and n are different in the ship and model.

To appreciate the error which may take place by assuming that skin resistance follows the laws of comparison, and also to illustrate how the reductions from model to ship are made—

In a ship, let $L = 345$ feet;

$\Delta = 93{,}000$ tons;

$S = 30{,}160$ square feet.

Suppose the model is run in fresh water, and has $\frac{1}{16}$ the linear dimensions of the ship, so that

$l = 21 \cdot 5$ feet

$s = 117 \cdot 9$ square feet.

Let the highest speed of the ship be 23·7 knots, that is, 40 feet per second; so that the corresponding speed for the model is $\frac{40}{\sqrt{16}} = 10$ feet per second. Assume surface of the model is equivalent to a varnished surface. From § 51, the resistance per square foot at 10 feet per second and a length of 21·5 feet is

$$\frac{0 \cdot 278 \times 20 + 0 \cdot 24 \times 1 \cdot 5}{21 \cdot 5} = 0 \cdot 276$$

and, therefore, the skin resistance of the model is 32·4 pounds.

For the ship in salt water, f (in knots) $= 0 \cdot 0089$ and $n = 1 \cdot 83$ (§ 52); hence

$$\text{skin resistance of ship} = 30{,}160 \times 0 \cdot 0089 \times (23 \cdot 7)^{1 \cdot 83}$$
$$= 87{,}800 \text{ pounds.}$$

By the law of comparison, the result is

$$32{\cdot}4 \times (16)^3 \times \frac{64}{62{\cdot}5} = 135{,}800$$

or 50 per cent. in excess of the true estimate.

§ 78. **Effect of Size on Economical Propulsion.**—Let Δ_1, Δ_2 be the displacements of similarly shaped ships; ${}_wR_1$ the wave-making resistance of Δ_1 at speed V_1. Then the wave-making resistance of Δ_2 is

$${}_wR_1 \frac{\Delta_2}{\Delta_1} \text{ at speed } V_1\left(\frac{\Delta_2}{\Delta_1}\right)^{\frac{1}{6}} = V_2.$$

If we assume that in the neighbourhood between V_1, V_2 the residuary resistance ${}_wR_1 \propto V^n$, the residuary resistance of the first at speed V_2 is ${}_wR_1 \left(\frac{\Delta_2}{\Delta_1}\right)^{\frac{n}{6}}$, so that the ratio of the resistance of the second to that of the first is $\left(\frac{\Delta_2}{\Delta_1}\right)^{1-\frac{n}{6}}$, and, therefore, the ratio of the resistance per ton is

$$\left(\frac{\Delta_2}{\Delta_1}\right)^{-\frac{n}{6}} \text{ or } \left(\frac{\Delta_1}{\Delta_2}\right)^{\frac{n}{6}}.$$

If the second be the larger vessel, then, provided n is positive, the resistance per ton is smaller in the larger ship, the ratio of the two being less the greater n. If $n = 0$, $R_w \propto \Delta$, and is independent of speed. If $n = 6$, R_w is independent of displacement, and the resistance per ton varies inversely as the displacement.

At *corresponding* speeds, the resistance varies as the displacement, and therefore the resistance per ton is the same in each.

Thus, so far as wave-making resistance is concerned, the resistance per ton, at a given speed, is less in the larger vessel.

If the skin friction varied as the square of the speed, the same laws would hold as for wave-making resistance. This, however, would over-estimate the skin resistance of the larger ship, but not to any great extent, if the vessels were not very dissimilar in size.

§ 79. **Illustrations of Cruisers.**—Sir William White[1] considers

[1] Presidential Address, Section G, British Association, 1899.

the case of a destroyer of 300 tons displacement, 212 feet long, with a maximum speed of 30 knots. Imagine a cruiser of similar form, 765 feet long, and, therefore, of 14,000 tons displacement. The ratio of the linear dimensions is 3·6, of displacement 47, of corresponding speeds 1·9, and of wave-making resistances at those speeds 47. To 12 knots in the destroyer corresponds 22·8 knots in the cruiser. The total resistance of the destroyer at 12 knots was 1·8 tons, so that for the cruiser at 22·8, neglecting friction correction, it would be 84·5. This would mean (at 22·8 knots) an effective horse-power of 13,250,[1] or an indicated horse-power of 26,500 (allowing a propulsive coefficient of 0·5). The frictional correction would probably reduce this to 25,000, making about 1·8 I.H.P. per ton. In the destroyer, at 22·8 knots, the total resistance is about 11 tons, giving an effective horse-power of 1725, which, allowing for a coefficient of propulsion of 0·57, gives an indicated horse-power of 3000, or about 1·00 I.H.P. per ton. This is about 5·5 times as great as in the cruiser.

Or again, at a speed of 30 knots in the large ship, the corresponding speed in the small ship is 15·8 knots. At that speed her resistance is 3·5 tons, and therefore that of the larger ship, at 30 knots, neglecting friction correction, is 165 tons nearly. The effective horse-power at 30 knots is therefore 34,000; and the indicated horse-power 68,000. Allowing for friction correction, this would probably be about 63,000, giving about 4·5 I.H.P. per ton. The destroyer at 30 knots has a resistance of 17½ tons, so that the E.H.P. is 3600, and the I.H.P. per ton is 6000 (coefficient of propulsion 0·6), giving 20 I.H.P. per ton, or 4·5 times as great as in the larger ship.

A further illustration is given by Sir William White. It refers to a number of typical cruisers which, although not exactly similar, possess many common characteristics. The numbers are "round."

[1] The effective horse-power is the horse-power necessary to propel the vessel, and is deduced from model experiments. The indicated horse-power is greater than the effective horse-power in that it includes losses in the engines, shafts, and in the propeller. The ratio of the effective horse-power to the indicated horse-power is called the *propulsive coefficient*.

	1	2	3	4	5
Length in feet	280	300	360	435	500
Beam	35	43	60	69	71
Mean draught	13	16¼	23¾	24½	26¼
Displacement in tons . .	1800	3400	7400	11000	14200
I.H.P. for 20 knots . . .	6000	9000	11000	14000	15500
I.H.P. per ton at 20 knots .	3·33	2·65	1·48	1·27	1·09

The data given are the results of actual trials, and include the efficiencies of the propelling machinery, propellers, and variation of form. The results, however, must be approximately correct.

The gain due to increased size is true at all speeds, but it is different at different speeds, because the index of velocity, n, is not constant (Fig. 69). Except in torpedo boats and destroyers, n usually increases with speed—for cruisers and battleships—and, consequently, more economical propulsion may be expected at high speeds than at low speeds. The table shows the progressive trial results of Cruisers No. 4 and 5—" rounded " results.

Speed.	I.H.P.		I.H.P. per ton.		Ratio.
	No. 4.	No. 5.	No. 4.	No. 5.	$\frac{\text{No. 5}}{\text{No. 4}}$
10	1500	1800	0·136	0·127	0·935
12	2500	3100	0·227	0·218	0·96
14	4000	5000	0·364	0·352	0·97
16	6000	7500	0·545	0·529	0·97
18	9000	11000	0·819	0·775	0·95
20	14000	15500	1·27	1·09	0·86
22	23000	23000	2·09	1·62	0·77

It will be noticed that up to 18 knots, there is a fairly constant proportion between the powers required for the two ships. The smaller vessel was designed for 20½ knots, and the larger for 22 knots. The I.H.P. per ton in the larger cruiser is 2·09.

In these cruisers, 40 per cent. of the displacement is required for the hull and fittings, so that the balance or "disposable weight" would be about 60 per cent.; say, 6600 tons for the smaller vessel, and 8500 tons for the larger, a gain of nearly 2000 tons for the latter.

§ 80. **Swift Torpedo Boats.**—The following table shows the advance in speed and power of torpedo boats and destroyers:—

Length.	Displacement.	I.H.P.	Speed.	I.H.P. per ton of displacement.
135	125	1500	23	12·0
150	150	2000	26	13·3
180	240	4000	27	16·7
200	300	6000	30	20·0
230	370	9000	32	24·5
Turbinia 100	44	2200	34	50·0

Sir William White summarizes the distinctive features of torpedo vessels:—

(1) The propelling apparatus is excessively light in proportion to maximum power developed. The engines run at a high rate of revolution—400 revolutions per minute—and water-tube boilers with forced draught are used. On trials each ton of propelling apparatus produces 45 indicated horse-power. As a comparison, a large modern cruiser with high-pressure steam and quick-running engines produces 12 indicated horse-power per ton engines and boilers; or only one-quarter that of the destroyer.

(2) A large percentage of the total weight is assigned to propeller apparatus. In a 30-knot destroyer nearly one-half the total weight is devoted to machinery and boilers. In the swiftest cruisers the corresponding allocation of weight is less than 20 per cent. of the displacement, and in the largest and fastest mail steamers it is about 20 to 25 per cent.

(3) The torpedo vessel carries a relatively small load of fuel, equipment, etc. In a 30-knot destroyer the speed trials are made with a load not exceeding 12 to 14 per cent. of the displacement. In a swift cruiser the corresponding load would be from 40 to 45 per cent., or proportionately three times as great.

(4) The hulls and fittings of torpedo vessels are exceedingly light in relation to the dimensions and horse-power.

In the *Turbinia*, the first experimental turbine boat built by Mr. Parsons, the boat was 100 feet long, 44½ tons displacement, and attained speeds of 33 to 34 knots in short runs. There are three shafts, each carrying three screws, each shaft rotating 2000 times a minute, and developing 2200 horse-power. The whole weight of machinery and boilers is 22 tons, so that 100 horse-power is produced for each ton of propelling apparatus.

§ 81. **Alteration of Form with a given Displacement.**—The heights of the waves formed, depending on the rate of variation of pressure in the stream line action, will depend on the form of the bow and stern. Experiments by Mr. W. Froude[1] enable us to judge, generally, how alteration of form and dimensions with a given displacement affect the resistance.

Mr. W. Froude experimented on four ships, the particulars being given in the following table:—

Ship.	Δ in tons.	Length in feet.				Extreme beam.	Mean draught.	Wetted surface.	β block coefficient.
		Entrance.	Parallel middle body.	Run.	Total.				
A	3980	144·0	72	144·0	360	37·2	16·25	18,660	0·64
B	,,	179·5	—	179·5	359	45·9	18·09	19,130	0·47
C	,,	154·5	—	154·5	309	49·4	19·32	17,810	0·47
D	,,	95·0	95	95·0	285	45·6	17·89	16,950	0·60

The total and residuary resistances, in tons, are given in the following table:—

[1] *Transactions of the Institution of Naval Architects*, 1874.

Knots.	Total.				Residuary.			
	A.	B.	C.	D.	A.	B.	C.	D.
12	4·99	4·79	4·64	5·51	1·075	0·76	0·89	1·94
16	10·7	9·64	9·76	21·25	4·05	2·82	3·45	15·22
20	33·2	17·32	20·35	—	23·35	7·05	10·8	—

Considering the residuary resistances, it will be noticed that A and B have the same length as well as displacement, but that B has a greater draught and extreme beam, but has no parallel body. The lines of B are, therefore, finer than A, and the residuary resistance throughout is less. It has a slightly greater wetted surface, but this is not sufficient to materially alter matters. Thus extreme width and draught when accompanied by finer lines are by no means conducive to increased wave making. On the contrary, the wave making appears to be much less.

Again, comparing B and C, it will be noticed that neither has any middle body, that C has the greater beam and draught, but is 50 feet shorter, and its lines are slightly fuller. The residuary resistance of C is, at all speeds, greater than B. The surface of C is, however, much less than that of B, with the result that below 16 knots the total resistance of B is slightly greater than C; whilst at 16 knots C has still the greater total resistance.

Finally, comparing B and D, it will be noticed that they have practically the same beam and draught as well as displacement, but that D is 74 feet shorter than B, and has a middle body. Its lines, therefore, are much fuller, and its resistance is markedly greater. Its surface is much less, so that at low speeds the difference is not so marked.

Sir William White pointed out in the discussion the desirability of having deeper vessels and finer lines with, of course, greater length of entrance and run, and greater extreme breadth.

In the ships taken, the resistance per ton is proportional to the resistance, since the displacement is the same in all.

The curves taken represent the curves of resistance. Curves of E.H.P. per ton of displacement will be steeper, but the

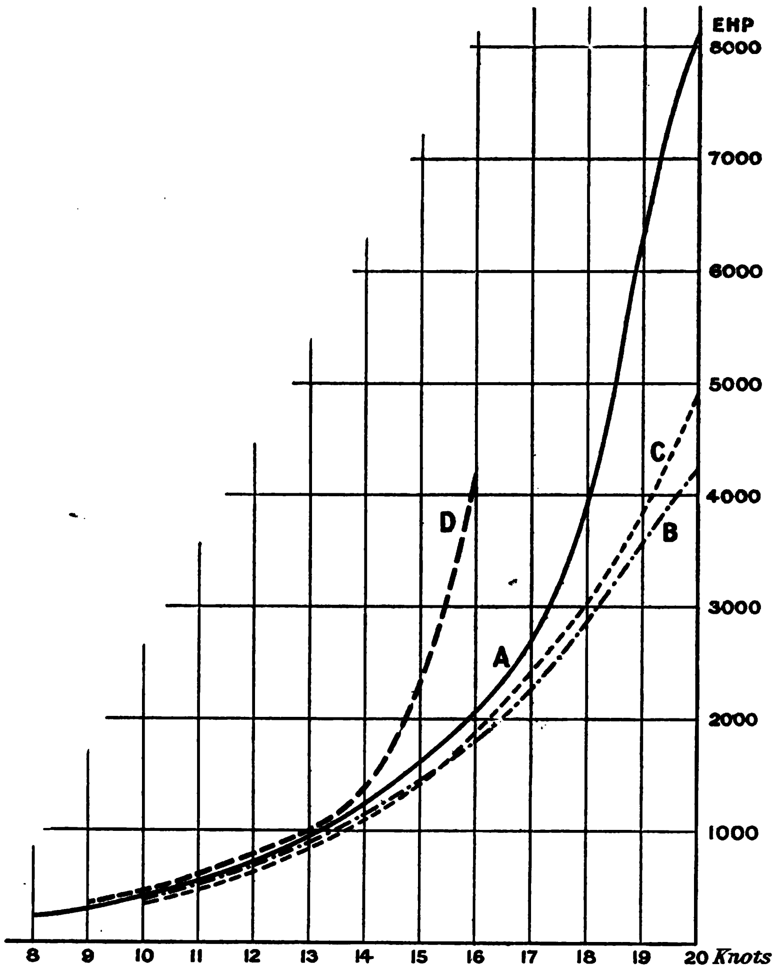

FIG. 79.

relative proportions at any one speed will be unaltered (Fig. 79). The following table gives the E.H.P. per ton for total and residuary resistance:—

Knots.	Total.				Residuary.			
	A.	B.	C.	D.	A.	B.	C.	D.
12	0·184	0·177	0·171	0·203	0·0397	0·0280	0·0328	0·0716
16	0·525	0·473	0·480	1·041	0·199	0·1388	0·1692	0·748
20	2·042	1·065	1·25	—	1·435	0·434	0·665	—

The total E.H.P., which is proportional to the E.H.P. per ton, is shown in Fig. 79. The curves B and C lie close together and cross at 15 knots. The curve A rises rapidly, and from 16 knots to 20 knots the ratio of E.H.P. is four. In curve D, the ratio of E.H.P., from 14 to 16 knots, is three.

§ 82. **Effect of Alteration of Draught and Displacement, the Beam and Length remaining unaltered.**—Mr. R. E. Froude shows resistance curves of long merchant-ships of the usual types, models of which have been tried at Torquay. The displacements, and also dimensions, are given in the following table, and they will be called E, F, G, H, in decreasing order of their displacement or draught:—

Ship.	Displacement in tons.	Length.	Extreme breadth.	Mean draught, including 9-ft. keels.	β block coefficient.	Ratio of displacement.	Ratio of draught.
		ft.	ft. in.	ft. in.			
E	5930	400	38 2	20 7	0·68	1·0	1·0
F	5390	,,	,,	19 1	0·67	0·908	0·92
G	4480	,,	,,	16 5	0·65	0·758	0·79
H	4090	,,	,,	15 4	0·64	0·689	0·73

The residuary resistance per ton in pounds is given in the following table:—

Speed.	E.	F.	G.	H.
11	1·02	1·12	1·12	1·21
14	2·72	2·78	2·70	2·68
17	7·06	6·74	6·16	6·03
20	18·00	18·20	16·30	15·20
23	30·05	27·80	24·70	23·58

From the table it will be noticed that wave-making resistance below 14 knots is less the deeper the draught, but above 14 knots it is greater the deeper the draught. So far as skin friction is concerned, the less the draught the less the skin friction, but the rate at which the skin resistance is decreased is much less than the rate at which the displacement is decreased. The reduction in surface is a small proportion because the bottom of the ship accounts for a large proportion of the surface; but the displacement varies in proportion to the immersion. Thus the skin resistance per ton increases as the draught increases. Thus, below 15 or 16 knots—the total resistance was not given—increased draught tends to increased economy, but above this speed this need not be true.

In the *Greyhound* (§ 60), it was pointed out that at 10 knots there was $10\frac{1}{2}$ per cent. less resistance under light than under deep draught, the reduction in wetted surface being about 8 per cent., and in displacement $19\frac{1}{4}$ per cent. In other words, at 10 knots the total resistance under a displacement of 1160 tons was 10,400 pounds, or 8·96 pounds per ton. Under a displacement of 938 tons it was 9320 pounds, or 9·95 pounds per ton.

§ 83. **Effect of Middle Body.**—It is important to discuss the effect on the economy of propulsion arising from increasing the length of the parallel middle body. It has been shown (§ 67) that, at a given speed, the wave-making resistance fluctuates about a constant mean value as the length, and therefore the displacement, is increased. Thus the resistance per ton of displacement will likewise fluctuate, but about an ever-decreasing value. The skin surface will increase in proportion to the additional length of middle body—that is, in proportion to the additional displacement—so that the skin resistance per ton of displacement will be, approximately, constant.

The curves of total resistance, given by Mr. William Froude,[1] in which a large number of points were plotted, have been given in Fig. 68.

The data contained in the following table have been taken from those curves:—

[1] *Transactions of the Institution of Naval Architects*, 1877.

Length in feet.	Displacement in tons.	Total resistance in tons.			Total resistance in pounds per ton.		
		Speed in knots.			Speed in knots.		
		6	10	14	6	10	14
160	1245	0·9	3·05	9·6	1·61	5·48	17·2
240	2419	1·55	4·60	14·7	1·44	4·26	13·67
320	3593	2·20	5·95	17·8	1·37	3·71	11·1
400	4767	2·65	7·10	18·5	1·24	3·33	8·69
480	5941	3·05	8·3	20·3	1·15	3·13	7·66

It will be noticed that the rate of reduction in total resistance in pounds per ton is more rapid the higher the speed.

§ 84. **Effect of Depth of Water on Resistance of Ships.**—It has been pointed out, in Chapter II., that wave-making resistance depends upon the rate at which the energy was draining away from the ship. The boat is followed by an ever-lengthening procession of waves; and the work required to draw a boat along—neglecting viscosity—is first equal to the work required to generate the procession of waves lengthening backwards behind the ship. In deep water, the rear of the procession moves forward at half the speed of the boat; relatively to the boat, it is moving backward at half the speed. In shallow water, or in confined channels, the stream lines must clearly be different to what they are in deep water, and so also must the variation of pressure along them, and therefore must the waves formed. Thus in shallow water, or if a confined channel is approached, we should expect an alteration in the wave system following the boat, and, therefore, in the wave-making resistance.

§ 85. **Experiments on a Canal.**—This alteration of resistance, and of the accompanying wave pattern, was markedly noticed by Mr. Scott Russell in 1835, and already noticed in Chapter II. On a canal in Scotland, a spirited horse attached to a boat took fright and ran off, dragging the boat with it, and it was then observed that the foaming stern surge, which used to devastate the banks, had ceased, and the vessel was carried on through the comparatively smooth water with a very greatly diminished resistance. Mr.

Scott Russell made accurate measurements for the pull in the tow-rope. The results of two experiments are given in the following table, and the results are plotted in Fig. 80:—

	Speed in miles per hour.	Resistance in pounds.	Tow-rope horse-power.
Boat, 4·6 tons displacement.	4·72 5·92 6·19 9·04 10·48	112 161 275 250 268½	1·41 4·12 4·54 6·03 7·50
Boat, 5·6 tons displacement.	6·19 7·57 8·52 9·04	250 500 400 280	4·12 10·1 9·1 6·75

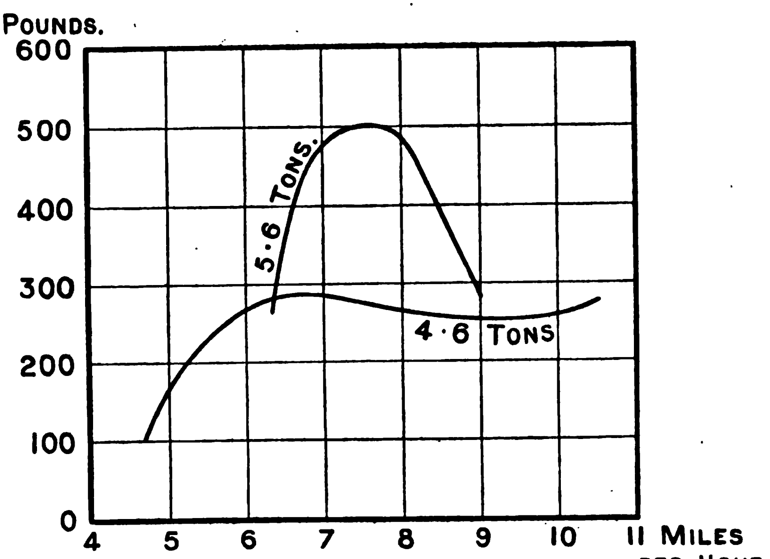

FIG. 80.

The average depth of water was about 4½ feet, so that the speed of a wave of translation would be 12 feet per second, or 7¼ miles per hour (§ 24). Fig. 80 shows the curves plotted. In the larger

boat the crest is at 8 miles an hour, and in the smaller boat at 7 miles an hour. The small boat decreases slowly beyond the crest of the curve, and slowly rises again. The larger boat descends rapidly, but if the speed were higher, the curve would rise again.

The reason for this variation is, at low speeds, the waves behind the boat are of short length. As the speed increases, so does the wave length and the heights, but the procession will not now lengthen astern at half the speed of the boat, but at a less rate. The procession, in fact, gets shorter, but consists of higher waves. As the speed still further increases, a critical stage is passed, the boat runs away, leaving no wave behind it, and experiencing no resistance whatever in the absence of skin friction and viscosity. The boat, in fact, rides on a wave of translation, whose speed depends on the depth of the channel and height of wave. It is, in fact, the most rapid wave that can travel in water of given depth. In the above wave it would be about 8 miles an hour, and when this speed is passed we get the diminution of wave-making resistance. In confined channels, and on account of the nearness of the boat to the bottom, the frictional resistance would probably increase at a higher rate.

§ 86. Captain Rasmussen's Experiments on Torpedo Boats.—Captain Rasmussen[1] has made some interesting experiments on the resistance of torpedo boats in shallow waters. The boat was 145 feet 6 inches long, 15 feet 6 inches beam, and 13·2 tons displacement. It was tried under four depths of water. In the paper, the curves of indicated horse-power are given, and these are shown in Fig. 81. The effect becomes more pronounced if curves of total resistance are plotted. In deducing the total resistance from the indicated horse-power, a propulsive coefficient of 0·55 has been taken. Fig. 82 shows the total resistance in tons plotted on a speed base. The depths of water are 15, 37½, 48, and 120 feet. The speeds of the waves of translation at these depths (§ 24) are 13·0, 20·6, 22·3, and 36·8 knots respectively.

In the two shallower depths, the speed of a wave of translation is reached, and the effect on both curves is most marked. For the smallest depth, the resistance increases very rapidly until

[1] *Transactions of the Institution of Naval Architects*, 1899.

the critical speed is reached; it then decreases slightly, and again commences to increase. The same things happen at the second

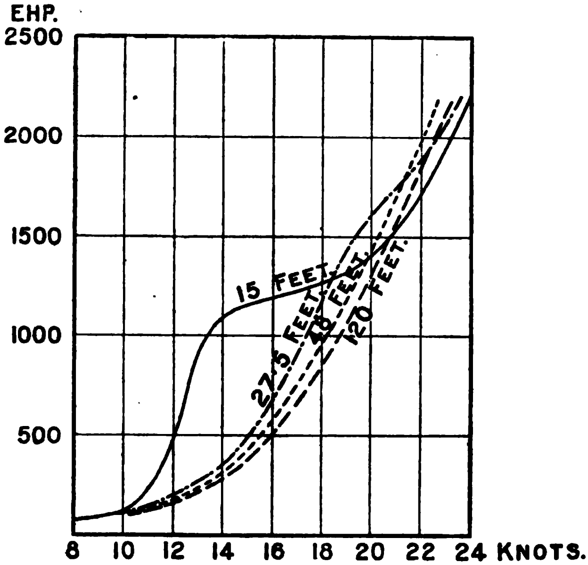

FIG. 81.

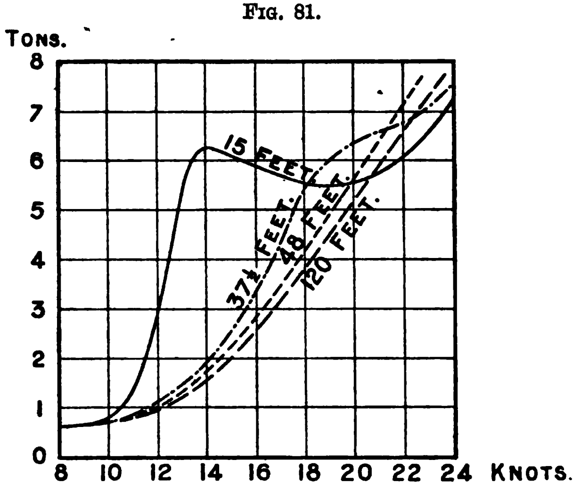

FIG. 82.

depth, but the phenomena are not so marked. At the third depth, the critical speed is 23·3, but the speeds are not great

enough to show the critical speed. These experiments show that the point of inflexion is much more marked the shallower the water, and the speed at which it takes place, being less the shallower the water. The system of waves which follow the boat is very unstable at speeds near the critical speed. Putting the rudder over might bring down the speed considerably; whilst it may become several knots greater at approximately the same power when the boat has been running a short time on a steady course. At the higher speeds (Fig. 82), it will be noticed that the indicated horse-power at a given speed might be less in shallow than in deep water; so that the speed, with a given horse-power, might be greater in shallow than in deep water. Thus, when developing 2200 I.H.P., the results are—

Speed in knots	24·1	23·8	22·8	23·6
Depth in feet	15	37½	48	120

When developing 1000 I.H.P., the results are—

Speed in knots	13·1	17·2	18·3	18·6
Depth in feet	15	37½	48	120

§ 87. **Mr. Denny's Experiments on the Effect of Depth.**[1]—The instability noticed by Captain Rasmussen (§ 86) has been observed by Mr. Archibald Denny, who made experiments on a model in the tank at Dumbarton. The curves in Fig. 83 refer to the total resistance of a model of a barge 200 feet long and 27 feet wide. The model was 12 feet long, beam 1·62 feet, draught 2 feet 3 inches, displacement in fresh water 274 pounds, block coefficient 0·837. The model was tried in 1·25 and 10 feet of water, the resistance curves being shown. The shaded part shows the extreme variations caused by instability at a depth of 1·25 feet. The change of trim is expressed in feet, and refers to the

[1] Discussion on Major Rota's paper, *Transactions of the Institution of Naval Architects*, 1900.

200-feet boat. On the same diagram is shown the still-water level, and the fall of stern and rise of bow measured from the still-water line at varying speeds, and in a depth of 1·25 feet.

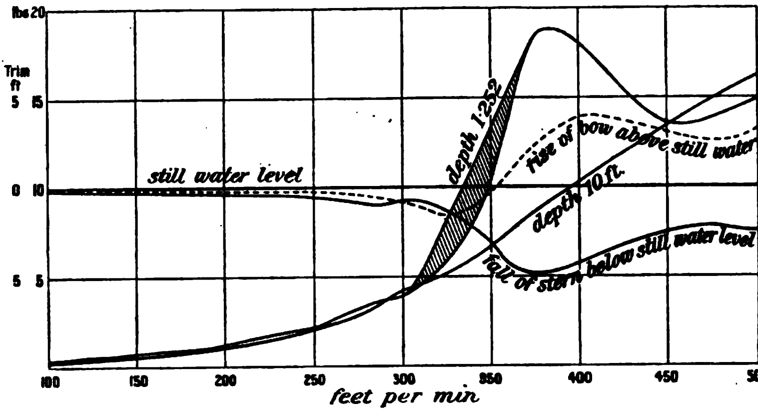

FIG. 83.

§ 88. **Major Rota's Experiments on the Effect of Depth on Resistance.**[1]—In the previous article, the curves of resistance on a speed base have been plotted for different depths of water. Major Rota, in the Italian experimental tank at Spezia, made experiments on five models, the first four being ordinary types, and the fifth a torpedo boat. The object of the experiments was to find the depth corresponding to a certain speed for which no further increase in depth causes an appreciable reduction in resistance.

The following are the dimensions of the models :—

Dimensions.	1	2	3	4	5
Length in feet . .	14·08	12·24	12·24	14·37	12·33
Breadth	2·59	2·13	1·87	1·52	1·35
Mean draught . .	0·92	0·68	0·68	0·53	0·39
Displacement pounds	1069·2	555·5	488·3	354·1	145·2
Block coefficient . .	0·51	0·50	0·50	0·49	0·43

[1] Major Guisseppe Rota, R.I.N., *Transactions of the Institution of Naval Architects*, 1900.

The variations in depth of water in the tank were obtained by using a solidly built wooden bottom, which could be varied in depth, the depth in the tank varying from 1¾ to 10 feet. The speed in the first four models varied between 3 and 8 feet per second, and the fifth model from 8 to 15 feet per second.

§ 89. **Models of Ordinary Type.**—The curves of resistance for the first four models, plotted on a speed base for different depths, were smooth curves. Fig. 84 shows the curves of resistance of model 3 at six depths. In Major Rota's paper the unit of

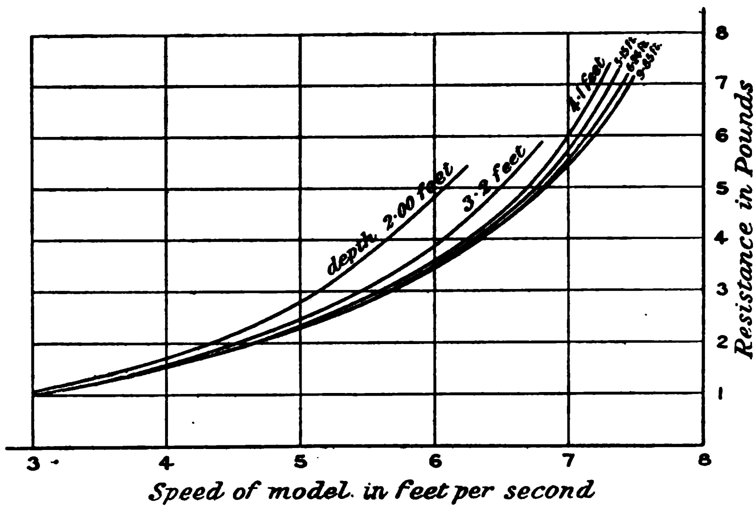

FIG. 84.

velocity was a metre per second, and the unit of force a kilogramme; for convenience, the curves have been redrawn and reduced to English measures. The smoothness might have been anticipated from the fact that in 2 feet of water the critical velocity is 8 feet per second, and in 10 feet of water 18 feet per second.

From Fig. 84 the resistance at a given speed—say 6 feet per second—but at different depths, may be found by reading the resistance. Thus (Fig. 85) a set of curves may be drawn, giving the resistance in pounds on a depth of water base at different

speeds. It will be noticed that all the curves are similar, and that after a certain depth the resistance remained sensibly constant. The dotted line represents the locus of minimum résistance. This

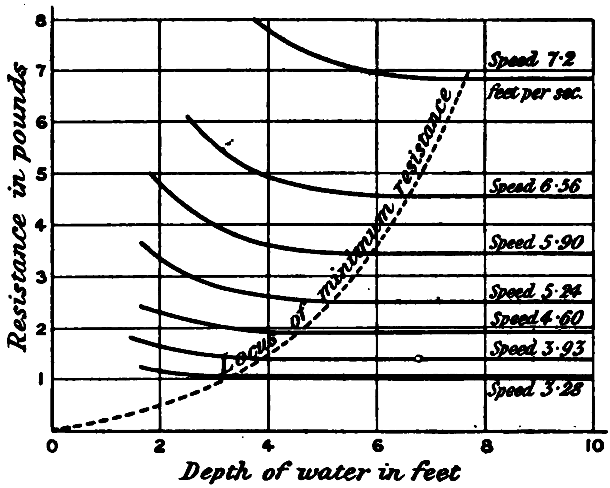

FIG. 85.

curve, apparently, has been drawn in by eye—no definite point, such as, for example, the point where the resistance is 1 per cent.

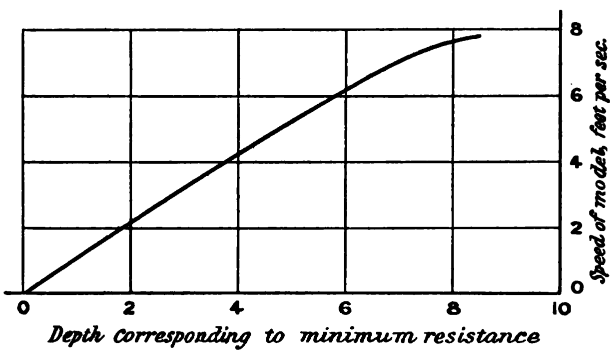

FIG. 86.

greater than in deep water, being taken. A third curve (Fig. 86) gives the speed corresponding to minimum depths of water at which the resistance might appreciably increase. This figure

refers to model 3, but the curve might be concave downwards or upwards, the curvature being gradual.

§ 90. **Model of a Torpedo Boat.**—In the torpedo boat, No. 5, the speeds in relation to the length are much greater than in the first four cases. Fig. 87 gives the curves of resistance on a speed base at six depths, ranging from 1 foot to 10 feet.

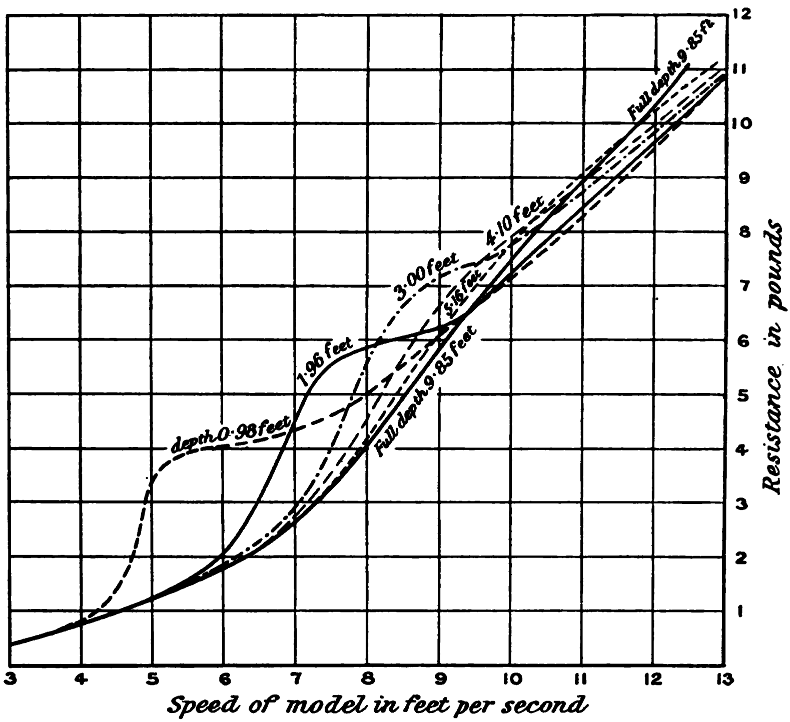

FIG. 87.

They are exactly similar in character to those deduced by Capt. Rasmussen (Fig. 82). At a certain speed, depending on the depth of water, the resistance curve rose rapidly, and then rounded off. The smaller the depth, the more rapid the "rise" and the sharper the "rounding off." The following table gives the critical speed, and the speed of a wave of translation for different depths:—

Depth in feet	0·98	1·96	3·00	4·10	5·16
Critical speed in feet per second	5·60	7·80	9·00	10·0	11·0
Speed of a wave of translation .	5·62	7·95	9·85	11·5	12·9

The higher the speed, the greater the difficulty in determining

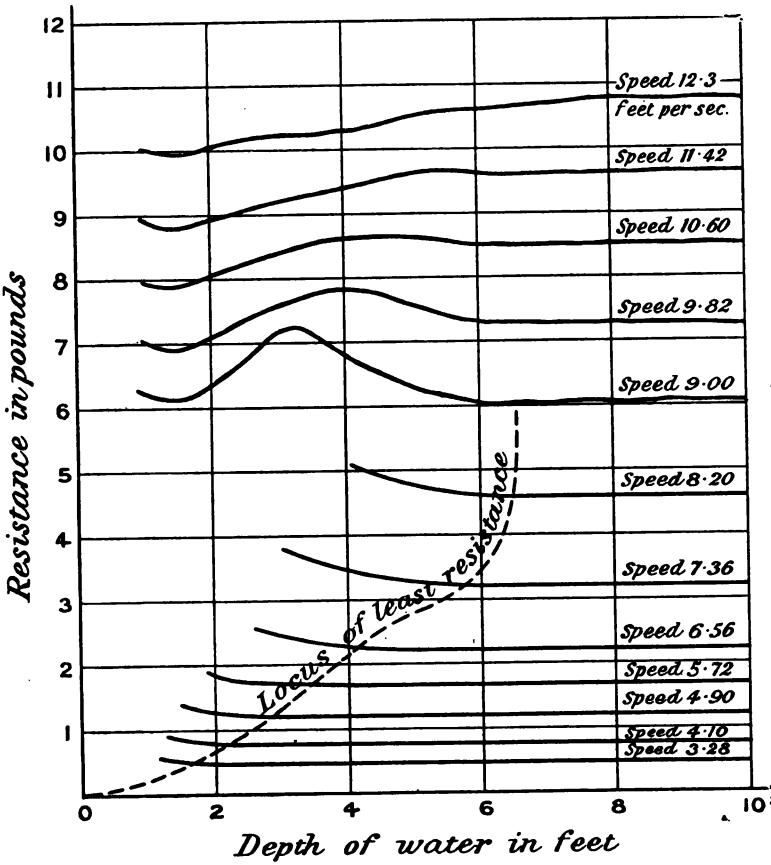

FIG. 88.

the critical velocity. Following the method already indicated for the first four boats, Fig. 88 gives the resistance in pounds on a

depth of water base at different speeds. Up to a speed of about 8 feet per second, the curves are similar to the first four models, but beyond this speed the curves of resistance are quite different. The minimum resistance is at about 18 inches of water, but, of course, it would be inadvisable to run in this depth of water, because a slight increase in depth—as shown by the curve corresponding to 9·30 feet per second—might cause a considerable increase in speed. In order to ensure consistent results the depth ought to be greater, although the resistance is not the least possible. It will be noticed that beyond 9·00 feet per second, the limiting depth is practically constant for all speeds.

Major Rota suggests that the decrease of the resistance at high speeds in shallow water may be found in the variation of trim,

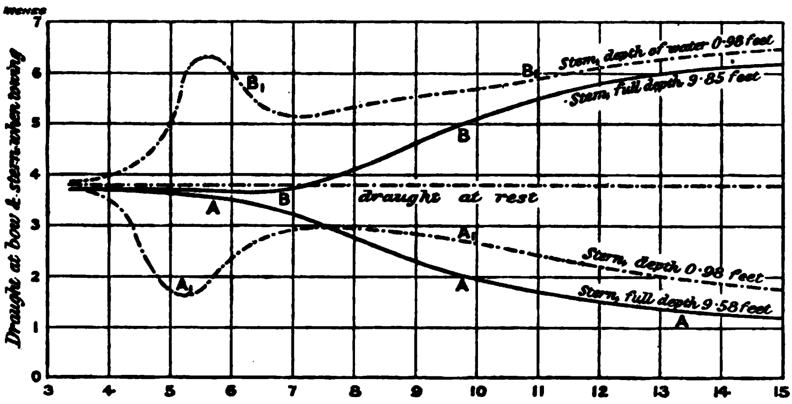

FIG. 89.

which is sensibly different from that observed in deep water. Fig. 89 shows the change of trim for model 5. The curve AA gives, with its ordinates, the variation of the aft perpendicular in deep water (10 feet) at different speeds; the curve BB, of the fore perpendicular. The curves A_1A_1, B_1B_1 give the variation of aft perpendicular and fore perpendicular for the least depth of water (1 foot). If the shapes of the curves A_1A_1, B_1B_1 be compared with the resistance curves in 1 foot of water (Fig. 87), it may be seen (1) that the maximum value of the

resistance on the first portion of the corresponding curves occur when the difference of trim is a maximum; (2) that the curve for a depth of 1 foot crosses the curve for 10 feet at a point corresponding to the same trim in both cases; (3) that the diminution of resistance at 1 foot depth as compared with that at 10 feet is coupled with a corresponding diminution of the difference of fore-and-aft immersion.

§ 91. **Extension to Actual Ships.**—Major Rota's results are based on model experiments. In extending to large ships, correction has to be made for skin friction. This would affect the resistance, and, no doubt, would affect, in some way, the shape of the curves shown in Figs. 83–88. Neglecting this correction, it will be interesting to express the results in terms of actual-sized ships. Reducing the first four models to a ship 400 feet long, and the fifth model to a destroyer 200 feet long, the proportions become—

	I.	II.	III.	IV.	V.
Length { model . . Length { boat . . .	14·03 feet 400 „	12·24 feet 400 „	12·24 feet 400 „	14·37 feet 400 „	12·33 feet 200 „
Beam	73·8 „	69·6 „	61·1 „	42·4 „	21·9 „
Draught	26·2 „	22·2 „	22·2 „	14·8 „	5·19 „
Displacement in tons	11,020	8660	7620	3430	276
Block coefficient (β)	0·51	0·50	0·50	0·49	0·43
Speed	11 to 24 knots				8 to 30 knots
Depth of water . .	50 to 320 feet				16 to 160 feet

Fig. 86 refers to model 3. The linear ratio between ship and model is 32·6, and the velocity ratio is 5·71. One foot per second n the model is 3·4 knots in the ship. Thus, to a sufficient

degree of approximation, the minimum depth for different speeds is as follows :—

Minimum depth in model	2	4	6
Minimum depth in ship	65·2	130·4	195·6
Speed in knots in ship	6·80	13·6	20·4

Thus the increase of depth with speed is about 10 feet per knot, and at 7 knots the minimum depth is 70 feet, about.

In the same way, at 7 knots, the minimum depths in models 1, 2, and 4 are 110, 95, and 40 feet respectively, the same addition being made per knot.

In model 5 (Fig. 88), taking a 200-feet destroyer, the linear ratio is 16·2, speed ratio 4·04, and 1 foot per second in model is 2·4 knots in destroyer. Thus, in the destroyer—

Minimum depth in feet .	26·6	45·5	60·0	90·0	104·0
Speed in knots	8·3	12·3	16·5	18·7	20·6

Up to 18 knots the increase of depth per knot is about 6 feet. At 20 knots, and at all higher speeds, the minimum depth may be taken as sensibly constant, and equal to 100 feet, and 6 feet must be deducted for each knot less than 20.

Sir William White, K.C.B., F.R.S.,[1] quotes the case of the *Blenheim* which, in 9 fathoms of water, developed 15,750 H.P., the revolutions being 92½, and the speed 20 knots. The wave phenomena were most striking and unusual. Later on, during the same trial, the ship ran for two hours in water of 22 to 36 fathoms in depth: the same power was developed; but, in consequence of the greater depth of water, the engines made 96½ revolutions, and the speed rose to 21½ knots.

[1] "Naval Architecture," p. 467.

§ **92. Estimates of Resistance and Horse-Power.**—In § 77 the wave-making resistance has been shown to follow the law—

$$R_w \propto \Delta^m V^n$$

in which $$m + \frac{n}{6} = 1.$$

The skin resistance follows the law (§ 51)—

$$R_s \propto \Delta^{\frac{2}{3}} V^{1 \cdot 83}.$$

Thus expressed generally, the total resistance must be of the form—

$$R = R_w + R_s = a\Delta^m V^n + b\Delta^{\frac{2}{3}} V^{1 \cdot 83}.$$

The value of n in the residuary resistance term may be found by logarithmic plotting, as shown in Fig. 50. But the "humps" and "hollows" which occur in the curve of resistance would make it impossible to get consistent results, as the value of n would vary from point to point.

Thus, from the total resistance curve, or the residuary resistance, the value of n can be determined.

§ **93. Illustrations.**—As regards the law of total resistance In a typical destroyer, 212 feet long (§ 80), up to 11 knots the total resistance varies nearly as the square of the speed; at 16 knots it has reached the cube; from 18 to 20 knots it varies as the 3·3 power. Then the index begins to diminish; at 22 knots it is 2·7; at 25 knots it has fallen to the square, and from thence to 30 knots it varies, practically, as does the frictional resistance.

The residual resistance varies as the square of the speed up to 11 knots, as the cube at 12½ to 13 knots, as the fourth power about 14½ knots, and at a higher rate than the fifth power at 18 knots. Then the index begins to fall, reaching the square at 24 knots, and falling still lower at higher speeds. It will be seen, therefore, that when this small vessel has been driven up to 24 knots by a large relative expenditure of power, further increments of speed are obtained with less proportionate additions to the power.

In the *Turbinia* (§ 80), from 10 to 15 knots the index is 3·0; at 16 knots, 2·0; and from 18·32 knots, 1·33.[1]

Frictional resistance is an important matter in all classes of ships, and at all speeds. In a destroyer (212 feet long, § 81) at 12 knots the friction with clean painted bottom represents 80 per cent. of the total resistance; at 16 knots, 70 per cent.; at 20 knots, a little less than 50 per cent.; and at 30 knots, 45 per cent. If the coefficient of friction were doubled and the maximum power developed with equal efficiency, a loss of speed of fully 4 knots would result.

In the cruiser of similar form, and 765 feet long (§ 81), the friction represents 90 per cent. at 12 knots, 85 per cent. at 16 knots, nearly 80 per cent. at 20 knots, and over 70 per cent. at 23 knots. If the coefficient of friction were doubled at 23 knots, and the corresponding power developed with equal efficiency, the loss of speed would approximate to 4 knots.

§ 94. Law of Resistance for Different Types.—The problem arises whether it is possible to estimate, with any degree of accuracy, the resistance of a new ship, or the horse-power to propel it at a given speed.

At low speeds, such as obtain in the mercantile marine, the wave-making resistance is a small percentage of the total (probably not more than 10 to 15 per cent.), and, in that case, the total resistance may be assumed to vary as V^2. It will also vary as $\Delta^{\frac{2}{3}}$, so that total resistance $\propto \Delta^{\frac{2}{3}}V^2$.

At higher speeds, such as obtain in mail steamers, the wave-making resistance is a considerable percentage of the total. The wave-making resistance alone will vary as V^2 when n has some value between 3 and 6. A common value may be taken as 4, in which case the wave-making resistance will vary as $\Delta^{\frac{1}{3}}$. As a general justification, the variation of pressure in stream line motion past a ship varies as the difference of the square of the velocity, and, therefore, the resistance varies as the fourth power of the velocity. It neglects, of course, the interference of the bow and stern waves, which might cause—due to the humps and hollows in

[1] Sir William White, Presidential Address, Section G, British Association, 1899.

the curve of residuary resistance—a considerable variation of the index at different ranges of speed. Thus, assuming skin friction to vary as the square of the speed, a law of resistance as applied to mail steamers and battleships may be expressed in the form

$$\text{total resistance} = a\Delta^{\frac{2}{3}}V^2 + b\Delta^{\frac{1}{3}}V^4.$$

In § 72, the variation in the fluctuating term in the residuary has been discussed. For similar shaped vessels—say, vessels of the same type—running at corresponding speeds, if L be the length of the boat, $\frac{V^2}{L}$ is constant and may be written c^2. Hence—

for small speeds, as in the mercantile marine—

$$R \propto V^2\Delta^{\frac{2}{3}} \propto c^2 L\Delta^{\frac{2}{3}} \propto c^2\Delta$$

for mail steamers and battleships—

$$R = \Delta(\alpha c^2 + \beta c^4)$$

from above formula.

It has also been pointed out that in the case of cruisers the fluctuations are large and fairly rapid, and in such cases it is not possible to predict any formula for the resistance. When the speed is very high, as in torpedo destroyers, c is large, but the fluctuations succeed each other at wider intervals. At these high speeds, the wave-making resistance again increases slowly with speed, and the law of resistance may be assumed to follow the law

$$R = aV^2\Delta^{\frac{2}{3}} = ac^2\Delta.$$

The resistance per ton is obtained by dividing by Δ. The following results are taken from Professor Cotterill's "Applied Mechanics," appendix 569.

Description.	Value of c (V in knots, L in feet).	$\frac{\text{Length of wave}}{\text{length of boat}}$	Number of waves per boat length.	Resistance in pounds per ton.
Mercantile marine .	$\frac{1}{2}$–$\frac{3}{4}$	0·14–0·32	7–3	Total = $a\Delta^{2/3}V^2$ where $a = 0·55–0·66$
Mail steamers . .	$\frac{3}{4}$–1	0·32–0·56	3–2	$8c^2(1 + c^2)$
Cruisers	1–$1\frac{1}{2}$	0·56–1·25	2–0·8	No formula
Torpedo boats . .	$1\frac{1}{2}$–$2\frac{1}{2}$	1·25–2·5	0·8–0·3	$30c^2$

If R be the total resistance in tons, the effective horse-power at 8 knots is

$$6{\cdot}89\text{VR}$$

in which, using the above results for mercantile marine—

$$\text{E.H.P.} = \frac{\Delta^{\frac{2}{3}}\text{V}^3}{500\text{–}600}$$

for mail steamers—

$$\text{E.H.P. per ton} = \frac{(c^2 + c^4)\text{V}}{45}$$

for torpedo boats—

$$\text{E.H.P. per ton} = \frac{\text{V}c^2}{11}.$$

CHAPTER VI

THEORETICAL CONSIDERATIONS AFFECTING THE PROPULSION OF SHIPS

§ 95. **Introduction.**—In Chapter III. the resistance due to eddy-making, and skin friction was discussed. In § 54 the equation of momentum was applied to the consideration of the "wake" following a plank.

In the first two chapters the fluid was assumed perfect, and the energy equation was applied—except (§§ 17, 18, 19) in the case of viscous flow between plates very near together.

Thus, in hydrodynamics, there are two principles: (1) the equation of energy; (2) the equation of momentum.

In applying the energy equation the changes of section are supposed to take place gradually, and the relation between p, v, z is given

$$\frac{p}{w}+\frac{v^2}{2g}+z=\text{constant}.$$

When eddies are present, as is invariably the case in practice, the momentum equation can only be applied.

It is important to distinguish between the energy equation and the momentum equation. The first is the effect of a force acting over a certain distance; the second the effect of a force acting over a certain interval of time. If P be the force, x the distance, w the mass, the expression for the first is

$$\mathrm{P}x=\tfrac{1}{2}mv^2$$

and of the second

$$\mathrm{P}t=mv.$$

When work is done on a number of bodies, or a number of particles of the same body, work is spent in two ways: (1) in increasing the kinetic energy of the system; (2) in overcoming mutual actions. In estimating each of these, no question of directions enters into the problem. In rigid bodies the value of the second is zero; but, in fluids, losses in friction and shock constitute a considerable fraction of the total energy necessary to produce a certain change. The energy thus necessary is to a great extent visible as rotating eddies or whirlpool of water, which ultimately reappears as heat.

In applying the second principle the forces act in a particular direction. "Change of momentum is proportional to the impressed force, and takes place in the direction in which the force acts." Thus it is only necessary to know the motion of a body, or particles of a body, in the direction of the force. If the accumulated momentum of a system has changed, a force must have operated to produce that change. In applying this principle, care must be taken to consider *all* the forces which act on the systems, and not merely those that do work. Such forces are, for example, the forces on the different elements of the hull of a vessel. In estimating the *energy* of a moving mass of water, the energy of the eddies, as well as the translational energy of the whole mass, must be included; but in estimating the momentum of the system, it is not necessary to include the eddies, because their momentum in every direction is zero. Thus, in many cases, the momentum may be used when the information or data do not permit the use of the energy equation; but, in some cases, either equation may equally well be used.

§ **96. Types of Propellers.**—There are three types of propellers—

(1) Jet Propeller.
(2) Paddle Wheel.
(3) Screw Propeller.

In the first type water is drawn into the ship through suitable orifices in the bottom, and projected sternwards by means of a pump in the vessel itself.

In the second type a large wheel rotates about a horizontal

axis, and by means of "paddles" sends a stream of water in an approximately sternward direction.

In the third type a blade, or number of blades fixed to a shaft at the stern of the ship, glides through the water at a slight angle with the direction of motion of the ship, and produces a circumferential and longitudinal motion of the water. The circumferential motion is produced by the engines in the ship, and the longitudinal motion propels the ship.

§ 97. **Jet Propeller.**—As already pointed out, water is drawn into the boat through suitable orifices in the bottom, and projected sternwards by means of a centrifugal pump.

Let V = velocity of ship;
v = velocity of exit relative to ship;
A = joint area of discharge orifices, in square feet.

The most efficient arrangement is one in which a scoop is used, that is, in which the entrance is so curved in the direction of motion that the water glides in the boat without shock.

Thus—

discharge in cubic feet per second $= Av$

change of velocity of water $=$ absolute velocity on leaving $= v - V$

change of momentum per second from water at rest

$$= mAv(v - V)$$
$$= \text{propulsive force} = T$$
$$= \text{resistance} = R$$

provided the boat is not being accelerated or retarded.

Useful work done $= VT = mAvV(v - V)$ foot-pounds per second.

Energy wasted in race $= \frac{1}{2}mAv(v^2 - V)^2$.

therefore, neglecting frictional losses,

work put in $= TV + \frac{1}{2}mAv(v^2 - V^2)$.
$= \frac{1}{2}mAv(v^2 - V^2)$.

If the boat be brought to rest by impressing the velocity V on the whole system, the absolute velocity impressed on the water is $(v - V)$, and therefore the increase of kinetic energy

in passing through the boat is, since mAv is the mass passing through, $\frac{1}{2}mAv(v^2 - V^2)$. The force necessary to hold the boat in place, under the conditions assumed, is $mAv(v - V)$.

Thus $\qquad T = mAv(v - V)$

work put in $\qquad = \frac{1}{2}mAv(v^2 - V^2) = T \times \frac{v + V}{2}$

efficiency of jet $\qquad = \frac{2V}{v + V}$.

This represents the efficiency when there is no waste of power except the inevitable one in the race.

For maximum efficiency, v and V must be nearly equal, and therefore $v - V$ must be as small as possible. With a given thrust and speed V, the smaller v is, the larger A must be. Other things being equal, the efficiency of a propeller is greater the greater the quantity of water on which it operates. For constructive purposes A cannot be greater than a certain quantity, and v is never less than 2V. If $v = 2V$, efficiency $= \frac{2}{3}$.

Altering A with a given v, alters the thrust in the same proportion as the work put in; but increasing v increases the thrust at a less rate than the energy wasted in the race.

The above efficiency assumes that the boat is supplied with a scoop, the water being allowed to flow practically unimpeded into the boat. If the boat has to steer equally efficiently when going ahead or astern, the water must be received into the ship through a hole in the bottom, in which case the water will have to be drawn into the boat by means of a pump. Thus the additional energy given to the water per second is, at least, $\frac{1}{2}mAvV^2$, so that the work put in is now—

$$\frac{1}{2}mAv(v^2 - V^2) + \frac{1}{2}mAvV^2 = \frac{1}{2}mAv^3 \text{ foot-pounds per second.}$$

The expression for the thrust will be the same as before, so that—

$$\text{efficiency of jet} = \frac{2V(v - V)}{v^2}.$$

If $v = 2V$, then in this case the feed and race losses are each equal to mAV^3.

Again, a further additional source of loss may be due to the

propelling machinery. If, when the water has been pumped into the boat, its velocity is suddenly changed—by the action of the pump—from V to v, the loss of head in shock is $\frac{(v - V)^2}{2g}$, and therefore the energy wasted per second is

$$wAv \cdot \frac{(v - V)^2}{2g}$$

which is the same as the race loss. Further, there are losses due to friction and bends, etc.

The propelling force is practically the same, whether the outlet nozzles are above or under water, so long as the outflow and the speed of the ship remain the same. If the orifices of discharge are placed under water, they will add greatly to the resistance of the ship; whilst if they are placed clear of the water, work must be done in raising the water above the general level of the sea. Probably the best position is to place them a little above water-line.

Finally, the presence of a large mass of water in the boat increases its displacement, and therefore its resistance; whilst the efficiency of the propelling machinery is invariably low.

If I.H.P. be the indicated horse-power of the engine, and η_e represent the mechanical efficiency of the engines, then the different efficiencies are made as follows:—

	Energy of feed available.	Not available.
η_{jet}	$\frac{2V}{v + V}$	$\frac{2V(v - V)}{v^2}$
η_{pump}	$\frac{\frac{1}{2}mAv(v^2 - V^2)}{550\eta_e \text{ I.H.P.}}$	$\frac{\frac{1}{2}mAv^3}{550\eta_e \text{ I.H.P.}}$
$\eta_{pump\ and\ jet}$	$\frac{mAvV(v - V)}{550\eta_e \text{ I.H.P.}}$	$\frac{mAvV(v - V)}{550\eta_e \text{ I.H.P.}}$
η_{total}	$\frac{mAvV(v - V)}{550 \text{ I.H.P.}}$	$\frac{mAvV(v - V)}{550 \text{ I.H.P.}}$

To show the advantage of the scoop—in the case of a boat (1875) in which the hole in the bottom was replaced by a scoop, the speed was increased from 7·87 to 8·12 knots with the same expenditure of power.

In 1881 Messrs. Thorneycroft constructed a jet-propelled boat with scoop and a second-class torpedo of practically the same dimensions. The torpedo boat was propelled by a screw. In the torpedo boat the efficiencies were—

$$\eta_{engine} = 0{\cdot}77, \qquad \eta_{screw} = 0{\cdot}65, \qquad \eta_{total} = 0{\cdot}5.$$

In the jet-propelled boat—

$$\eta_{engine} = 0{\cdot}77, \qquad \eta_{pump} = 0{\cdot}46, \qquad \eta_{jet} = 0{\cdot}71,$$
$$\eta_{pump\ and\ jet} = 0{\cdot}38, \qquad \eta_{total} = 0{\cdot}254.$$

Thus the screw boat has twice the efficiency of a jet-propelled boat, so that they are very wasteful.

For special purposes where moderate speed is sufficient, where simplicity is an important element, or where an ordinary propeller may get fouled, or where great manœuvring power is required—as in a steam lifeboat—this mode of propulsion may have advantages which justify its adoption. The manœuvring power is usually obtained by multiple discharge nozzles so situated and so adjustable that the reaction may be directed in any line, and the boat propelled in either direction or transversely, or turned in either direction as on a pivot.

§ 98. **Paddle Wheel.**—As already pointed out, a paddle wheel is a large wheel rotating about a horizontal axis, transverse to the longitudinal section, and by means of "paddles" sends a stream of water in an approximate sternward direction.

Let V = velocity of the ship;
v = velocity of paddle floats relative to the ship;
A = combined area of race on both sides;
Q = Av = quantity of water operated upon per second, that is, the quantity of water which has the velocity increased from V to v relative to the ship.

Assuming the water is initially at rest, and that the water in

the race is projected sternwards with the velocity of the paddles, then

$$T = \text{momentum accumulated per second}$$
$$= mQ(v - V) = mQsv^2 = mAsv^2$$

in which $$s = \frac{v - V}{v}.$$

s is termed the *apparent slip*.

The useful work per second = RV foot-pounds.

If the velocity with which the propeller acts against the resistance is v, then, neglecting frictional and other losses, work put in per second $= Rv$,

and efficiency $$= \frac{V}{v} = 1 - s.$$

The total work lost $= R(v - V) = mQ(v - V)^2$. The work lost in the race is $\frac{1}{2}mQ(v - V)^2$, so that the work lost otherwise than in the race is also $\frac{1}{2}mQ(v - V)^2$. This loss is due to the shock, since it has been assumed that the velocity is increased from V to v relatively to the boat. With radial floats there would be loss in agitation of the water, which may be reduced in paddle wheels using feathering floats. But the loss quoted above is common to all paddles. In addition there are losses due to frictional resistances, and also principally because other motions are impressed on the water than those in a sternward motion, which increases the work done by the engines without increasing the thrust.

The paddle is much more efficient than the jet propeller, because the area A, and therefore the quantity of water operated on per second, is much greater, so that the same thrust may be obtained with a given V with a less value of v; and this notwithstanding that the paddle causes losses in giving transverse motions which do not exist in the jet propeller.

The average slip of a paddle is about 20 to 30 per cent. under favourable conditions.

In the case of the jet and paddle, which may be termed direct-acting propellers, the results may be expressed in a general way as follows:—

Let T = thrust resistance of boat if not being accelerated or retarded;

S = absolute sternward velocity, in feet per second, impressed on the water;

Q = the quantity of water operated on per second.

Then, considering the energy in the jet—

$$\text{energy put in} = \tfrac{1}{2}m\{(V + S)^2 - V^2\}Q$$
$$= mS\left(V + \frac{S}{2}\right)Q.$$

$$\text{The useful work} = mS.V.Q$$

$$\therefore \text{efficiency} = \frac{V}{V + \dfrac{S}{2}}.$$

In the paddle—

$$\text{useful work} = mS.V.Q$$
$$\text{work put in} = mS(V + S)Q$$
$$\text{efficiency} = \frac{V}{V + S}.$$

In the first case the velocity of the water is gradually increased from V to V + S, the mean velocity against which the propeller acts being $\frac{V + V + S}{2} = V + \frac{S}{2}$; in the second case it is V + S.

Screw Propeller

§ **99. Primary Considerations.**—Let us consider, in the first place, how a screw propeller operates, and consider any small element of the blade. When motion takes place, the element traces out a spiral on a cylinder having a radius equal to the distance of the element from the centre of the shaft, but the pitch of that spiral will be less than the pitch of the element on account of "slip." Suppose the cylinder and spiral are developed.

Let AB (Fig. 90) be the element considered, the direction of motion of the ship being as shown.

Also let λ = pitch of element;
n = revolutions per second;
r = radius of element AB;
v_o = speed of screw = $n\lambda$;
V = speed of ship.

Make OL = $2\pi rn$ = circumferential velocity of element, and LF = v_o, so that OE represents the path and velocity described by the centre of the element.

If there were no slip, the element AB would move edgewise along OF, and, in the absence of friction, would neither experience resistance nor deliver thrust. Actually, however, it moves parallel to itself along the line OE, so that it is, in fact, a plate moving through the water in a direction which makes an angle FOE with its own plane, the angle FOE being called the *slip* angle, which is invariably small, rarely exceeding 10°. The plate, therefore, will experience a normal and tangential resistance both of which may be resolved into a longitudinal and circumferential component. The circumferential component is obtained by the turning movement supplied by the engines; the longitudinal force thus introduced is the thrust of the screw.

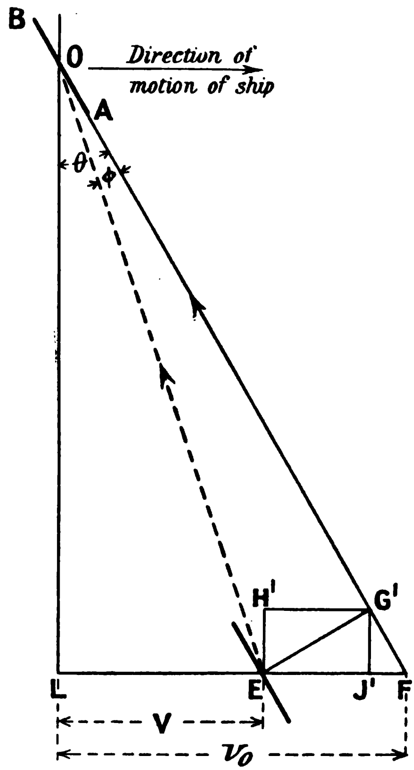

FIG. 90.

Thus the whole solution of a screw propeller resolves itself into finding the longitudinal and circumferential forces on the

plate. These may be deduced probably from direct experimental data, or the forces may be calculated from the subsequent motions of the water. The former is the method adopted by Mr. William Froude; the latter by Professors Rankine and Greenhill, and others.

§ **100. Theories of the Screw Propeller.**—In considering the different theories which have been propounded of the screw propeller, it is best to consider the element AB.

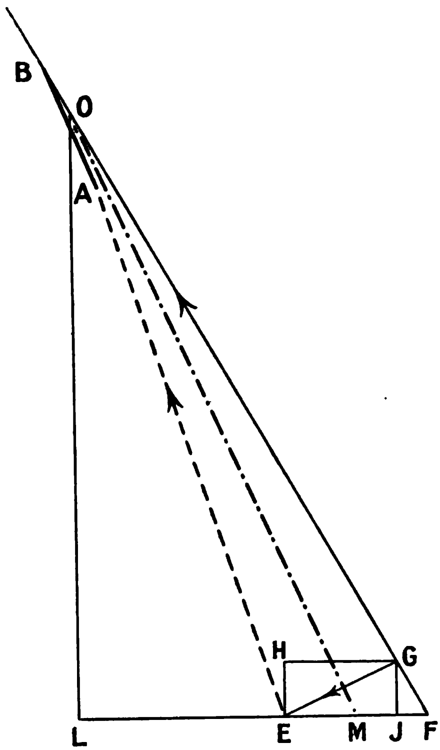

FIG. 91.

(1) *Screw with gaining pitch.*—In Fig. 91, suppose the element AB, instead of having a constant pitch angle FOL, has a pitch angle EOL at the leading edge A, and an angle FOL at the following edge B. As the element moves through the water, it will clearly both receive and discharge the water without shock, and will experience no resistance, but will impress a certain velocity on the water. The motion may be reduced to a steady motion by imagining the element to be fixed, and the water to flow towards O along EO with a velocity represented by EO. It will be received by the plate without shock, and will be discharged in the direction FO. Moreover, since the velocity with which the water glides over the plate is—in the absence of friction—unaltered, the velocity of discharge—assuming the pressure to remain constant—

must be equal to OE, and may therefore be represented by OG. The change of motion is represented in magnitude and direction by GE, and this, likewise, will represent the absolute velocity impressed upon the water if the element move along the direction OE through still water. Thus the sternward velocity impressed on the water is represented by GH, and the rotary velocity by HE.

In estimating the forces acting on the element, imagine the element to be of small radial thickness, and neglect the effect of centrifugal action. The race corresponding to the element will consist of an annulus of water which moves spirally astern with an absolute velocity given in magnitude and direction by GE. Let a be the area of this race measured transversely. The absolute sternward velocity impressed on the water is represented by GH; hence, if GJ be drawn perpendicular to the direction of the ship, the sternward velocity in the race, relative to the ship, is represented by LJ, and the quantity of water operated on per second (q) is

$$q = a \,.\, \mathrm{LJ}.$$

If T be the thrust of the element in a sternward direction, then

$$\mathrm{T} = mq \,.\, \mathrm{JE}$$

being the change of momentum per second. If L be the couple on the shaft, then

$$\mathrm{L} = mq \,.\, \mathrm{GJ} \,.\, r$$

being the change of angular momentum per second.

The work got out

$$= \mathrm{T} \,.\, \mathrm{LE}.$$

The work put in

$$\begin{aligned} \mathrm{L} \times \text{angular velocity} &= mq \,.\, \mathrm{GJ} \,.\, r \times 2\pi n \\ &= mq \,.\, \mathrm{GJ} \,.\, \mathrm{OL} \\ &= mq\mathrm{EJ} \tan \mathrm{GEJ} \,.\, \mathrm{OL} \\ &= mq\mathrm{EJ} \,.\, \mathrm{OL} \cot \mathrm{OML} \\ &= \mathrm{T} \,.\, \mathrm{LM} \end{aligned}$$

where OM is a line drawn from O perpendicular to EG to meet LF in M. If LE be taken to represent the leading pitch, and LF the following, then LM may be called the *mean effective pitch.* Thus, the loss of work

$$= \mathrm{T} \,.\, \mathrm{EM}$$
$$= \tfrac{1}{2}mq \,.\, \mathrm{GE}^2.$$

Thus, in a screw with gaining pitch, the only loss is the inevitable one in the race.

(2) *Screw with constant pitch.*—Consider the element to have a constant pitch. In Fig. 90, if the element be reduced to rest, as in the first case, the water will impinge on the plate in the direction EO. Before the water reaches the plate it has a component velocity G′O in the direction of the plate, and a component velocity G′E normal to the plate. Professor Rankine assumed that the tangential component remained unaltered; that is, that the particles of water were projected normally from the plate. The absolute velocity impressed upon the water will be represented by G′E, the sternward component by J′E, and the circumferential component by G′J′. Using the previous notation—

$$q = a \,.\, \mathrm{LJ}'$$

and $$\mathrm{T} = mq \,.\, \mathrm{EJ}' = ma \,.\, \mathrm{LJ}' \,.\, \mathrm{EJ}'$$

$$\mathrm{L} = mq \,.\, \mathrm{G'J'}.$$

Work put in $$= ma \,.\, \mathrm{LJ}' \,.\, \mathrm{G'J'} \,.\, \mathrm{OL}$$
$$= \mathrm{T} \,.\, \mathrm{LF}$$

since $$\frac{\mathrm{G'J'}}{\mathrm{EJ'}} = \frac{\mathrm{LF}}{\mathrm{OL}}.$$

The work lost is

$$\begin{aligned} \mathrm{T}(\mathrm{LF} - \mathrm{LE}) &= \mathrm{T} \,.\, \mathrm{EF} \\ &= (mq)\mathrm{EJ}' \,.\, \mathrm{EF} \\ &= mq \,.\, \mathrm{EG}'^2 \end{aligned}$$

that is, twice the energy in the race. The loss, therefore, is double that for a gaining pitch. The reason is that, in addition to the loss in the race, there is a loss in shock, because the normal component of the velocity G′E is, on the assumption made, destroyed.

In both these theories, the thrust—neglecting centrifugal action—is represented entirely by a change of momentum, there being supposed to be no variation of pressure in the race. The force necessary to cause this sternward momentum is obtained by the rotation of an oblique paddle, which impresses upon the water a rotary as well as a sternward velocity. Each particle thus moves in a spiral path, and the water contracts as it passes through the propeller.

(3) *Professor Greenhill's theory.*—If the theory given for an element be extended to the whole race, it is obvious that, due to centrifugal action, there must be a variation of pressure in the race. It is therefore possible that the thrust is due, not entirely to change of momentum, but also, in part, to a change of pressure. A centrifugal pump, for example, receives water at one pressure and discharges it at a higher one; and the pressure at discharge can be varied by suitably arranging the proportions of the wheel. In a screw, also, it may be that part of the thrust is due to pressure.

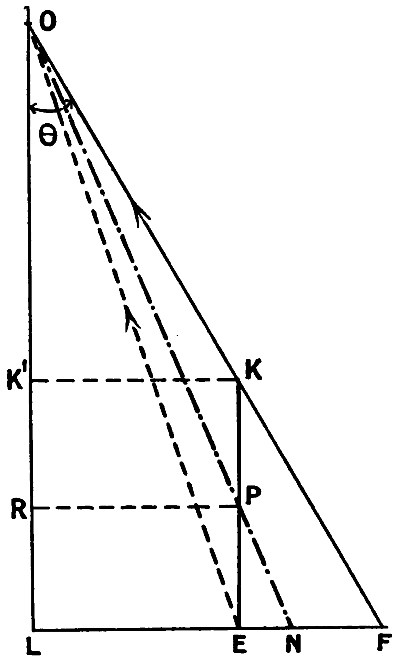

FIG. 92.

Consider an extreme case. Imagine a screw to advance in a pipe with closed ends which it accurately fits, the pipe being filled with water. If the pipe be always full, there must be no contraction of the stream, so that the pressure cannot give any sternward momentum to the water. No backward escape of the water being possible, the water behind the screw will be left rotating in planes perpendicular to the axis, and the angular momentum

generated in the water is the mechanical equivalent of the couple required to turn the propeller, whilst the thrust of the screw must be wholly represented by a pressure head generated in the pipe. In that case, whether the screw have a gaining pitch or constant pitch, the absolute velocity impressed on the water is KE (Fig. 92), since the water enters the plate with velocity EO, and leaves it with velocity KO. In a gaining pitch there are no shock losses, and, reducing the problem to steady motion as before, the change of kinetic energy must be equal to the gain of pressure, so that

$$\text{gain of pressure} \quad = \tfrac{1}{2}m(\text{OE}^2 - \text{OK}^2) = \tfrac{1}{2}m(\text{OL}^2 - \text{OK}'^2)$$

in which P is the middle point of EK, K′ and R are the projections of K and P on OL,

$$\text{thrust} \quad = ma\,.\,\text{OR}\,.\,\text{KE} = mq\,.\,\frac{\text{OR}\,.\,\text{KE}}{\text{LE}} = \text{T}$$

$$\text{useful work} \quad = \text{T}\,.\,\text{LE} = mq\,.\,\text{OR}\,.\,\text{KE}$$

$$\text{work put in} \quad = mq\,.\,\text{KE}\,.\,\text{OL} = \text{T}\,.\,\frac{\text{LE}}{\text{OR}}\,.\,\text{OL} = \text{T}\,.\,\text{LN}$$

in which LN may be called the mean effective pitch;

$$\text{therefore the efficiency} \quad = \frac{\text{LE}}{\text{LN}}.$$

The loss of work is

$$\begin{aligned}\text{T}(\text{LN} - \text{LE}) = \text{T}\,.\,\text{EN} &= mq\,.\,\frac{\text{OR}\,.\,\text{KE}}{\text{LE}}\,.\,\text{EN} \\ &= mq\,.\,\text{KE}\,.\,\frac{\text{KE}}{2\text{EN}}\,.\,\text{EN} \\ &= \tfrac{1}{2}mq\text{KE}^2\end{aligned}$$

which represents the loss of energy in the race.

With a constant pitch

$$\text{work put in} \quad = mq\text{EK}\,.\,\text{OL}$$

$$\begin{aligned}\text{useful work} \quad &= mq(\text{KE}\,.\,\text{OL} - \text{EK}^2) \\ &= mq\,.\,\text{EK}\,.\,\text{OK}'\end{aligned}$$

$$\text{thrust} \quad = \frac{mq\,.\,\text{EK}\,.\,\text{OK}'}{\text{LE}} = mq\,.\,\text{EK}\cot\theta$$

$$\text{efficiency} \quad = \frac{\text{OK}'}{\text{OL}} = \frac{\text{LE}}{\text{LF}}.$$

Otherwise the circumferential force is mq . EK, and the resultant is normal to the plane, so that, resolving, T $= mq$. EK cot θ, as before.

(4) *Professor Rankine's theory.*[1]—Since the velocity of the water is increased in some way in passing through the screw, the sectional area of the column must contract in a similar manner to the stream of water issuing from an orifice in a thin plate. Whether the contraction takes place wholly in the screw, or partially in and partially before the screw, is a matter for later discussion.

The assumption made in Rankine's theory has been pointed out in § 99. For purposes of calculation it is usually assumed that the sectional area of the race is the same as that of the screw disc.

In addition, it is assumed that not only all the water on which the screw operates is projected sternwards in a column not differing greatly in diameter from the screw itself, but that also each element projects its corresponding annular cylindrical column, so that the whole cylindrical race may be imagined divided into a number of concentric hollow cylinders, each having a sternward and rotatory motion which, in general, will be different for each cylinder, so that the cylinders slide through each other and rotate within each other. Moreover, it is assumed (1) that the form and dimensions of the blades are such as are consistent with the completeness of the column; (2) that the supply of water is sufficient; (3) that the screw works in undisturbed water; (4) that centrifugal action is not so great as to break up the column. So long as these conditions are satisfied, the form of the blade is immaterial, except in so far as a variation in form might cause a variation in surface, and there is skin friction, which, however, will be neglected.

§ 101. Estimation of Thrust and Turning Moment on an Element.—Consider a screw of constant pitch λ (§ 99), and let the diagram (Fig. 90) refer to a column of radius r, and thickness dr, the pitch angle of the element forming it being θ. As before, let v_0 (= LF). be the speed of the screw, and V (= LE) the speed of the ship,

[1] This theory was published in 1865.

and let v be the sternward velocity of the water in the annulus considered relative to the ship (that is = LJ′), whilst ω represents the angular velocity of the *water* (so that ωr = H′E).

The transverse area of the annulus is

$$2\pi r dr$$

therefore the total quantity of water operated on per second

$$= Q = 2\pi \int v r dr \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the screw did not revolve, this would have been

$$2\pi \int V r dr$$

so that the quantity is increased by the operation of the screw.

Moreover, the sternward force necessary to produce the momentum in the race

$$\begin{aligned} = T &= \int m(2\pi r dr \,.\, v)(v - V) \\ &= 2\pi m \int v(v - V) r dr \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

and the couple necessary to generate the rotary motion of the column, and which is exerted by the engines on the screw shaft

$$\begin{aligned} &= \text{moment of momentum of fluid} \\ &= L \\ &= \int (m \,.\, 2\pi r dv \,.\, v) \omega r \,.\, r \\ &= 2\pi m \int \omega r^3 v dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3) \end{aligned}$$

Now, to express v and ω in terms r, v_o, V and λ, it must be remembered that for each element v_o, V and λ are the same. Let the slip be s, so that

$$s = \frac{v_o - V}{v_o} \left(= \frac{EF}{LF} \text{ in Fig. 91} \right)$$

and let $$q = \cot \theta = \frac{2\pi r}{\lambda} \left(= \frac{OL}{LF} \right)$$

so that s is the same for each element, whilst q increases as the radius increases.

Then the absolute sternward velocity of the water

$$= H'G' = v - V = EG' \cos \theta = EF \cos^2 \theta = \frac{s v_o q^2}{1 + q^2}$$

and $\quad v = (v_o - sv_o) + sv_o \dfrac{q^2}{1+q^2} = v_o\left(1 - \dfrac{s}{1+q^2}\right);$

the rotary velocity

$$= \text{H}'\text{E} = \omega r = \text{EG}' \sin\theta = \text{EF}\sin\theta\cos\theta = sv_o\frac{q}{1+q^2}.$$

The absolute velocity is represented in magnitude and direction by G′E, so that the magnitude is

$$sv_o\frac{q}{\sqrt{1+q^2}}$$

and its pitch angle is $\left(\frac{\pi}{2} - \theta\right)$, *i.e.* $\tan^{-1}q$.

Moreover, since q is the variable—

$$dr = \frac{\lambda}{2\pi}dq.$$

If suffix (1) refer to the outer boundary, so that when $r = r_1$, $q = q_1$; then, neglecting the boss, the limits of r are r_1 and o, and of q, q_1 and o.

If A be the total area of the race, then

$$\text{A} = \pi r_1^2 = \frac{\lambda^2}{4\pi}q_1^2.$$

Thus—

$$\begin{aligned}
\text{Q} = 2\pi\int_o^{r_1} vrdr &= 2\pi\int_o^{q_1} v_o\left(1 - \frac{s}{1+q^2}\right)\frac{q\lambda}{2\pi}\cdot\frac{\lambda}{2\pi}dq \\
&= \frac{\lambda^2}{2\pi}v_o\int_o^{q_1}\left(q - s\cdot\frac{q}{1+q^2}\right)dq \\
&= \frac{\lambda^2}{2\pi}v_o\left\{\frac{q_1^2}{2} - \frac{s}{2}\log_e(1+q_1^2)\right\} \\
&= \text{A}v_o\left\{1 - \frac{s}{q_1^2}\log_e(1+q_1^2)\right\} \quad . \quad . \quad (4)
\end{aligned}$$

$$\begin{aligned}
\text{T} &= 2\pi m\int_o^{r_1} v(v - \text{V})rdr \\
&= 2\pi m\int_o^{q_1} v_o\left(1 - \frac{s}{1+q^2}\right)\cdot\frac{sv_oq^2}{1+q^2}\cdot\frac{\lambda q}{2\pi}\cdot\frac{\lambda}{2\pi}dq \\
&= \frac{ms\lambda^2v_o^2}{2\pi}\int_o^{q_1}\left\{\frac{q}{1+q^2} - \frac{sq}{(1+q^2)^2}\right\}q^2dq.
\end{aligned}$$

Integrating by parts, the result is

$$= \frac{ms\lambda^2 v_o^2}{2\pi}\left\{\frac{sq_1^2}{2(1+q_1^2)} - \frac{1+s}{2}\log_e(1+q_1^2) + \frac{q_1^2}{2}\right\}$$

$$= mAsv_o^2\left[1 - \frac{\log_e(1+q_1^2)}{q_1^2} - s\left\{\frac{\log_e(1+q_1^2)}{q_1^2} - \frac{1}{1+q_1^2}\right\}\right] \quad (5)$$

The couple

$$L = 2\pi m\int_o^{q_1}\frac{sv_o q}{1+q^2}\frac{\lambda^2}{4\pi^2}q^2 v_o\left(1 - \frac{s}{1+q^2}\right)\frac{\lambda}{2\pi}dq$$

$$= \frac{ms\lambda^3 v_o^2}{4\pi^2}\int_o^{q_1}\left\{\frac{q}{1+q^2} - \frac{sq}{(1+q^2)^2}\right\}q^2 dq$$

$$= \frac{\lambda}{2\pi}T, \text{ by comparison of integrals} \quad . \quad . \quad . \quad . \quad (6)$$

The work put in per revolution is $L \times 2\pi = \lambda T$. The work get out $= T\lambda(1 - s)$.

$$\therefore \text{efficiency} = 1 - s.$$

This might have been anticipated, because the slip is constant for all elements—the pitch being constant—and as it has been proved for one element, it is true for the whole blade.

Since L and T depend on density of the liquid, and also the resistance of the ship, the design of the propeller will be independent of the density of the fluid.

§ 102. **Effect of Centrifugal Action.**—The above expression for T merely expresses the change of momentum in a sternward direction, and will, therefore, represent the thrust of the screw, provided there is no alteration of pressure in the interior of the column. But, on account of the rotary motion of the water, the pressure is diminished as the centre of the column is approached, and this diminution of pressure causes the thrust to be less than that given by T. To find what the diminution of pressure is, let p be the pressure—either radially, circumferentially, or in a sternward direction—at radius r; then

$$dp = m\omega^2 . rdr$$

$$= m\omega^2 r^2 \frac{dr}{r}$$

$$= ms^2v_o^2 \frac{q^2}{(1+q^2)^2}\frac{dq}{q}$$

$$= ms^2v_o^2 \frac{q}{(1+q^2)^2} dq$$

$$\text{therefore } [p] = \left[- \frac{ms^2v_o^2}{2(1+q^2)} \right].$$

When $r = r_1$, $q = q_1$; and let $p = p_1$, so that

$$p_1 - p = \frac{1}{2} ms^2v_o^2 \left\{ \frac{1}{1+q^2} - \frac{1}{1+q_1^2} \right\}.$$

If p_1 is the pressure at the fringe of the race in still water, and π the still water press in front of the propeller at the point in front of the point where the pressure is p, then

$$\pi - p = \frac{1}{2} ms^2v_o^2 \left(\frac{1}{1+q^2} - \frac{1}{1+q_1^2} \right).$$

The resultant thrust, therefore, upon an annular cylinder is the thrust due to the sternward momentum minus the sucking action—that is to say

$$\left\{ mv(v - \mathrm{V}) - \frac{1}{2} ms^2v_o^2 \left(\frac{1}{1+q^2} - \frac{1}{1+q_1^2} \right) \right\} 2\pi r dr.$$

If this expression be equated to zero, it will give the elementary cylinder with zero thrust. Within this, the cylinder is subjected to negative thrust, that is, the water must be conceived to press against the back force of the screw instead of on the front. If the value of v be substituted in terms of q, the equation becomes

$$\frac{1}{1+q^2} \left\{ \frac{s}{2} + \frac{sq^2}{1+q^2} - q^2 \right\} = \frac{s}{2(1+q_1^2)}$$

a quadratic in q^2.

The loss of thrust due to centrifugal action is

$$\rho = \int_o^{q_1} 2\pi r dr \frac{1}{2} ms^2v_o^2 \left(\frac{1}{1+q^2} - \frac{1}{1+q_1^2} \right)$$

$$= \frac{ms^2v_o^2\lambda^2}{4\pi} \int_o^{q_1} \left(\frac{q}{1+q^2} - \frac{q}{1+q_1^2} \right) dq$$

$$= \frac{1}{2} m\mathrm{A}s^2v_o^2 \left\{ \frac{\log_\epsilon(1+q_1^2)}{q_1^2} - \frac{1}{1+q_1^2} \right\}.$$

This reduction of pressure might produce two effects: (1) the turning moment on the screw shaft may be altered in magnitude; (2) if the reduced pressure in the rear of the screw cause a flow of water in front of the screw, the reduced pressure might extend to the hull of the vessel, and so increase the resistance.

§ 103. **Numerical Example.**—The following table shows the results calculated on Rankine's theory (1) with the ship stationary, (2) with a slip of 25 per cent.:—

Speed of screw = 1000 feet per minute = 16·67 feet per second. Circumference = 2 × pitch.	V = 0, ship stationary.	V = 12·5 screw marins δ=0·25.
1. Mean velocity of water through screw in feet per second	10	15
2. (Absolute) mean sternward velocity impressed upon water	10	2·5
3. Percentage supply due to action of screw	100	16·7
4. (Absolute) sternward velocity at surface of column	13·34	3·34
5. (Absolute) rotatory velocity at surface of column	6·67	1·67
6. (Relative) direction of motion of surface particles	26°	6°
7. Diminution of pressure at centre of column in inches of water	41·5	2·6
8. Diameter of that portion of column for which net thrust is zero	0·434	0·175
9. Thrust of screw in pounds per square foot by Rankine's formula	215 fresh 221 salt	74 fresh 76 salt
10. Maximum deduction for centrifugal action	53·8 fresh 55·3 salt	3·36 fresh 3·45 salt
11. Net thrust	165·7 salt	72·55 salt
12. Area of equivalent to give same thrust at same speed and slip	0·3 with deduction, 0·4 without deduction	0·525 with deduction, 0·55 without deduction

It will thus be seen that under ordinary conditions the maximum reduction of pressure is about 2 or 3 inches of water.

Professor Rankine discussed the question of the alteration of thrust and turning moment due to friction. This will be discussed later (§ 108).

§ 104. **Professor Greenhill's Theory.**[1]—In Rankine's theory, or in the theory of a propeller having gaining pitch (§ 100), the thrust of the propeller—with the correction for centrifugal action—is principally due to change of sternward momentum. Under ordinary conditions the correction for centrifugal action (see lines

[1] *Transactions of the Institution of Naval Architects*, 1888.

9 and 10 in table just given) is negligible compared to the thrust due to sternward momentum.

As an alternative theory, Professor Greenhill suggests a case in which the thrust is due entirely to pressure. The main principles have been discussed under section (3) of § 100, in which the expressions were obtained for the useful work, and work put in for a screw with gaining pitch, and a screw with constant pitch.

Let Fig. 92 refer to a screw of constant pitch, in which the rotation is the same as in Fig. 90. As no sternward velocity is impressed upon the water, the rotary velocity is represented by KE. And

$$\begin{aligned} \mathrm{EK} &= \mathrm{EF} \cdot \frac{\mathrm{OL}}{\mathrm{LF}} \\ &= sv_o \cdot \frac{2\pi rn}{v_o} \\ &= 2\pi nsr. \end{aligned}$$

Thus the water behind the propeller must, under the conditions assumed, rotate as a solid mass with an angular velocity $2\pi ns$, that is, at ns revolutions per second.

Moreover, since the quantity of water operated on per second is AV where A is the area of the pipe,

$$\begin{aligned} \text{the couple on the shaft} &= \mathrm{L} \\ &= \text{angular momentum per second} \\ &= m\mathrm{AV}(2\pi ns) \, . \, k^2 \text{ and } k^2 = \frac{r^2}{2} \\ &= m\mathrm{A}^2 nv_o(1 - s)s. \end{aligned}$$

Also, if $d\mathrm{Q}$ be the circumferential force at any radius, and $d\mathrm{T}$ the thrust, clearly (Fig. 92)—

$$d\mathrm{T} = d\mathrm{Q} \cot \theta = \frac{2\pi rn}{v_o} d\mathrm{Q} = \frac{2\pi n}{v_o} \, . \, d\mathrm{L}$$

or

$$\begin{aligned} \mathrm{T} &= \frac{2\pi n}{v_o} \mathrm{L} \\ &= 2\pi mn^2 \mathrm{A}^2 (1 - s)s. \end{aligned}$$

$$\begin{aligned} \text{The work put in per second} &= \mathrm{L} \times 2\pi n \\ &= m\mathrm{A}^2 nv_o s(1 - s) \, . \, 2\pi n \\ &= 2\pi mn^2 \mathrm{A}^2 v_o s(1 - s). \end{aligned}$$

The useful work per second $= \mathrm{TV}$

$$= \mathrm{T}v_0(1 - s)$$
$$= 2\pi mn^2\mathrm{A}s(1 - s)v_0(1 - s).$$

Hence the efficiency is $(1 - s)$, as in all cases having a constant pitch.

The energy lost per second $= 2\pi n \,.\, \mathrm{L} - \mathrm{TV}$

$$= \mathrm{T}(v_0 - \mathrm{V})$$
$$= \mathrm{T} \,.\, sv_0$$
$$= 2\pi ns\mathrm{L}$$
$$= m\mathrm{AV}(2\pi ns)^2 \,.\, k^2$$

= twice the kinetic energy in the race.

Of this loss, one-half represents the energy communicated to the revolving motion of the wake of the screw, the other half being the loss due to the shock in the rotatory velocity being suddenly impressed on the water. The latter may be prevented, section (3), § 100, by having a screw with gaining pitch.

§ **105. Equilibrium of Pressures.**—To find the pressure at any point in the race, at radius r

$$\text{intensity of pressure} = \frac{d\mathrm{T}}{2\pi r dr} = \frac{2\pi n \,.\, d\mathrm{L}}{v_0 \,.\, 2\pi r dr}$$
$$= \frac{n}{v_0 r dr}(m\mathrm{V} \,.\, 2\pi r dr \,.\, 2\pi nsr \,.\, r)$$
$$= 4\pi^2 mn^2 s(1 - s)r^2.$$

This will not only give the axial pressure, but it must also, if equilibrium exist, be the radial pressure. Now, the distribution of radial pressure can be otherwise determined. Thus, if dp be the change of pressure corresponding to a change of radius dr, then

$$dp = m\omega^2 r dr = m \cdot 4\pi^2 n^2 s^2 \cdot r dr$$
$$\therefore [p] = 2\pi^2 n^2 s^2 \,[r^2]\,;$$

or, if the excess pressure is zero when $r = 0$, at radius r

$$\text{excess pressure} \qquad = 2m\pi^2 n^2 s^2 v^2.$$

Clearly, for equilibrium, these two expressions for the excess pressure must be equal, in which case

$$s = 0{\cdot}67.$$

For any other value of s

$$\frac{\text{axial pressure}}{\text{radial pressure}} = \frac{2(1-s)}{s} \text{ for all radii.}$$

For different slips

s	0	0·05	0·1	0·2	0·5	0·67	0·8	1
Ratio =	∞	38	18	8	2	1	0·5	0

It would thus appear that equilibrium cannot exist; in other words, the simple motion assumed by Professor Greenhill cannot exist unless the slip is 67 per cent. At moderate slip, the effect of centrifugal action is feeble, the pressure produced by the screw blade greatly preponderating; at large slips, the reverse holds. The effect at moderate slip ratios is that the great pressure at the circumference of the pipe would, if the supposed motion were artificially produced, at once be transmitted to the centre, the velocity of the water through the centre of the screw would be checked, and the sectional area it occupies correspondingly increased, thus squeezing out the outer layers. The water at the circumference is thus accelerated and its pressure falls, this process going on until there is equilibrium due to the thrust of the screw blade, and that due to centrifugal action. The table shows that even with a closed tube very little of the thrust at the outer elements can be due to pressure, the greater part being represented by change of momentum. At large slip ratios water is accelerated near the centre and retarded near the tips. When the screw is stationary, a jet of water is projected through the centre of the screw, while a counter-current exists at the circumference.

Thus a blade of uniform pitch, working in a closed pipe, must, at moderate slips, throw back water at the circumference just as it does when working in open water, where a variation of pressure at the tip is impossible. In a closed pipe, since there is no net contraction, there is a reversed action at the centre so that the mean velocity and, consequently, the total momentum remains unaltered. The thrust of the screw, therefore, notwithstanding

this circulation, must still be represented by a pressure head, but the variation of pressure will be different to the results already obtained.

§ 106. **Motion in Open Water.**—So far the motion has been in a closed pipe. Screw propellers operate in open water. According to Professor Greenhill's theory there is a pressure which increases outwards, and a circulation is produced except at one particular slip. In open water, therefore, there would be a discontinuity of pressure in the fringe of the race. But in practice there can be no discontinuity of pressure, the pressure at the fringe of the race is practically that in still water. Consequently it follows that the thrust at the tips must be entirely due to the backward momentum imparted to the race. But this need not be true for the body of the race. In Rankine's theory the assumption is made for the body of the race. Professor Greenhill makes the assumption that the sternward velocity of the race is the same throughout, and equal to that at the fringe. This will affect the distribution of pressure in the race, and will affect the angular velocity.

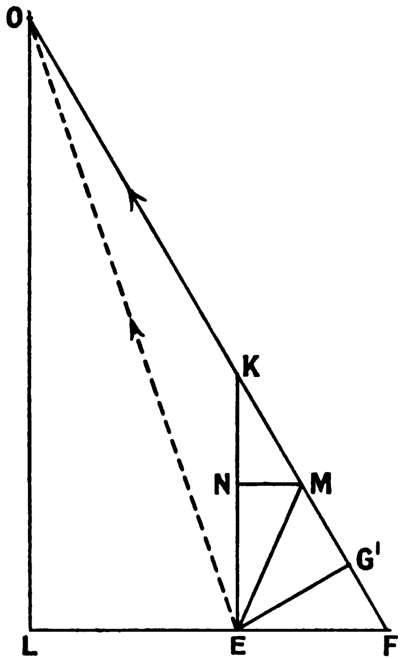

Fig. 93.

Suppose (Fig. 93) the motion impressed upon the water is represented by ME, this is an intermediate case between Rankine's theory (EG′) and the theory just considered. The rotatory velocity

$$\begin{aligned} &= \omega r = \mathrm{EN} = \mathrm{EK} - \mathrm{KN} \\ &= (\mathrm{EF} - \mathrm{NM}) \cot\theta \\ &= [sv_o - (v - \overline{1 - s}\, v_o)] \cot\theta \\ &= r \,.\, 2\pi n\left(1 - \frac{v}{v_o}\right) \end{aligned}$$

in which v, as before, represents the sternward velocity in the cylindrical element considered relatively to the ship. If v vary, so will ω; but if the wake have a uniform sternward velocity impressed upon it, then the angular velocity will likewise be constant.

Now, whether the thrust be due to change of sternward momentum or to increased pressure, the relation between dT and dL for any cylindrical element is given by

$$d\mathrm{T} = \frac{2\pi n}{v_o} d\mathrm{L}$$

in which dL represents the angular momentum of the cylinder considered, and must therefore—assuming the area of the race to be that of the screw disc, as in Rankine's theory—be equal to

$$mv \times 2\pi r dr \times \omega r \times r$$

so that—
$$d\mathrm{T} = \frac{8m\pi^3 . n^2 r^3 v(v_o - v)}{v_o^2} dr.$$

The part of the thrust represented by the change of sternward momentum

$$mv(v - \mathrm{V}) . 2\pi r dr.$$

The difference, therefore, represents that part due to pressure, and by dividing that difference by $2\pi r . dr$, the intensity of axial thrust at radius r is

$$4m\pi^2 n^2 r^2 \frac{v(v_o - v)}{v_o^2} - mv(v - \mathrm{V}).$$

On Rankine's theory this is zero, and

$$q^2(v_o - v) - (v - \mathrm{V}) = 0$$

$$\therefore v = v_o\left(1 - \frac{s}{1 + q^2}\right) \text{ as before.}$$

On the assumption made by Professor Greenhill, $v = \mathrm{V}$, so that the second term is zero. Also if the sternward velocity is the same throughout the race, so that the angular velocity is constant, being equal to that at the fringe, on the assumption that at the tips the thrust is due entirely to change of momentum, then

$$v = v_o\left(1 - \frac{s}{1 + q_1^2}\right).$$

The pressure at radius v is then

$$-mv(v-V)\left(1-\frac{r^2}{r_1^2}\right)$$

which is negative, as might have been anticipated. If this diminution extend forward of the propeller, it will cause augmentation of resistance of the hull. Moreover, the quantity of water—assuming the race to be of the same area as the screw disc—is increased in the ratio v to V. Hence

$$L = mA^2nv\left(1-\frac{v}{v_o}\right)$$

and
$$T = \frac{2\pi nL}{v_o}.$$

Mr. R. E. Froude's theory.[1]—Mr. Froude points out that the difference between Professor Rankine's and Professor Greenhill's theories is, that in the former case the propeller is regarded as directly imparting a spiral motion to the column of water on which it operates, whilst in the latter it is regarded as directly imparting a rotary motion only.

The momentum due to imparting the spiral motion contemplated by Rankine maintains a corresponding spiral force, which may be resolved into an axial component or thrust, and a rotary component or turning moment; in Greenhill's treatment, the propeller imparts to the column a purely rotary motion, the momentum due to which involves an axial component or thrust which, in the absence of change of axial speed of the water, can be balanced by only a difference of water pressure before and behind the propeller. Rankine's assumption implies a contraction of column within the length of the propeller, which becomes more and more unreal the shorter the lengths of the blades relatively to the diameter. In the limiting case, where the length of the blades is infinitesimal compared to the diameter, Rankine's assumption cannot hold, and Greenhill's must be considered.

In both theories the particular characteristics of the propellers are not considered, so that the given quantity of water and the speed

[1] This theory was developed in two papers published in 1889, and re-published in 1892 in the *Transactions of the Institution of Naval Architects.*

imparted to it, and the corresponding reactions and work expenditure, are at once determinable. In other words, the speeds assumed and the conclusions deduced from them are independent of the question of the particular mechanism by which those speeds may be supposed to be imparted. The propeller is, in fact, treated as an expedient whereby undisturbed water is somehow implicated in a machine, and forcibly compelled to assume certain motions, and so discharged. The thrust or turning is merely represented by the dynamic equivalent of the motions impressed. The increase of speed is supposed, in the case of a screw of uniform pitch, to take place suddenly, which results in a certain loss of head. The question arises whether, even in the case of a screw of uniform pitch, this suddenly impressed velocity actually takes place; or whether the water may not be accelerated beyond the confines of the propeller.

§ **107. Mechanical Illustrations—Actuator.**—To illustrate this point, Mr. R. E. Froude considers the case of a propeller in which the water is directed sternwards. Imagine a straight tube (Fig. 94)

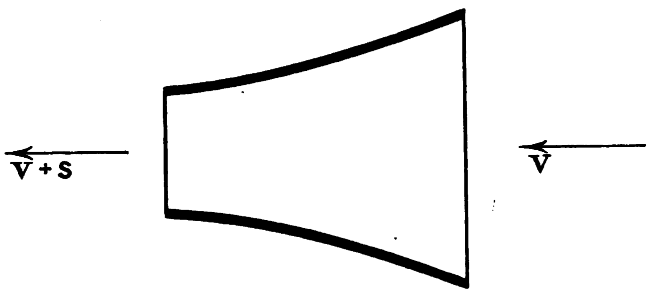

Fig. 94.

tapering gradually—the precise form does not matter—from the area A to an area $\frac{AV}{V + S}$, V being the velocity with which the tube moves through the water, and S the absolute sternward velocity impressed upon the water. If the tube be brought to rest, the velocity at the entrance would be V, and at the exit V + S (Fig. 94). Neglecting frictional resistance, the gain of pressure will be

$$w \frac{(V + S)^2 - V^2}{2g}$$

and, if the pressure of the water in front of the tube and to the rear of the tube be the same, there must have been some cause which gives the rise of pressure. Mr. Froude imagines that this gain of pressure may be effected by an infinite series of thin pistons fed into the tube at the large end in close succession, contracting in diameter as they pass along the tube, and abolished as they emerge at the small end, each actuated by a small force in the direction of flow, sufficient, at each point, to just restore the head lost. The pressure will then be everywhere uniform, so that the tube would be inoperative and might be discarded, and if the tube of pistons be supposed advancing at speed V into still water, it will represent an ideal propeller with purely internal acceleration. The aggregate thrust on all the pistons will be represented by

$$T = \frac{w}{g} AVS$$

and the work expended will be

$$wAV\frac{(V + S)^2 - V^2}{2g} = T\left(V + \frac{S}{2}\right)$$

as before (§ 98).

Now imagine the tube to be again restored, and suppose the entire restitution of pressure takes place at one point of the tube. Then no matter where the "actuator" is placed, the work expended by the actuator will be the same—being the quantity of water pumped by the actuator multiplied by the head against which it is pumped; and, moreover, the net thrust developed will be the same, namely, equal to the product of the quantity of water into the speed. The net thrust is, of course, the thrust on the actuator minus the forward pressure on the walls; and these two together are equal to the change of momentum per second. If the actuator be placed at the entrance of the tube, the pressure on the walls of the tube will be everywhere greater than that of the surrounding water, so that the pressure on the actuator would be greater than the net thrust. On the other hand, if the actuator be placed at the exit of the tube, the pressure on the walls would be less than the surrounding water, and so the pressure on the actuator would be less than the net thrust. At any intermediate position of the

actuator, the pressure on the walls in front of the actuator would be less, and in the rear of the actuator less, than in still water. For some position of the actuator, the negative pressure on the walls before the actuator will exactly balance the total pressure after the actuator, and the net thrust is then equal to the forward force on the actuator.

If the velocity at this point is v_1, then

$$w\frac{\mathrm{AVS}}{g} = \frac{\mathrm{AV}}{v_1}w\frac{(\mathrm{V}+\mathrm{S})^2 - \mathrm{V}^2}{2g}$$

whence

$$v_1 = \mathrm{V} + \frac{\mathrm{S}}{2}.$$

Thus, so long as the actuator is situated at a point where the speed is $\mathrm{V} + \frac{\mathrm{S}}{2}$, the thrust of the system is exactly that of the actuator alone, and so far as the force in the fore and aft direction is concerned, the walls are inoperative. The total work expended and the net thrust delivered will be represented by a sudden change of pressure, the velocity being increased by an amount $\frac{\mathrm{S}}{2}$ before and after the change. This advancing surface of pressure would thus give the desired results, and at no point would any sudden increase of velocity occur. There would be no loss of head in shock, neither would there be the contraction in the small length of the propeller.

But the tube is not yet necessarily inoperative. It still performs the duty of resisting the surplus of external pressures before the pro-

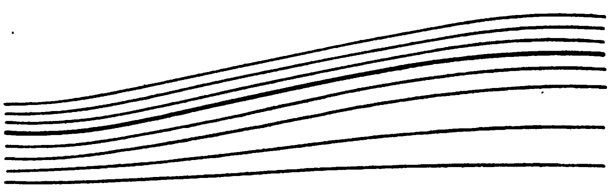

FIG. 95.

peller, and of internal pressures behind it; and in order to remove the tube, and so make the final step to the conception of the surface

of change of pressure advancing into open water, to render the tube wholly inoperative, it must be imagined that the curvature of the streams supplies the external and internal pressures before and after the actuator (Fig. 95). At the edge of the actuator there can be no sudden change of pressure, so that, as before, at the edge of the actuator the streams may be supposed to receive a finite change of velocity instead of its equivalent of pressure. The effect on the streams will shade off, by intermediate gradations, from one of change of speed at the edge, to one of change of pressure at a finite distance within it. Thus the water layers suffer contraction, but the argument points to the possibility of a plane of change of pressure rather than of velocity.

This refers to a sternward motion. In a screw there is rotary motion which causes a reduction of pressure in the race, and which will necessitate certain conditions to be satisfied before equilibrium in the race can exist between the pressures due to thrust and centrifugal action.

Mr. R. E. Froude[1] takes into acccount the effect of rotary motion in the tube. He takes the case already quoted, and later the radial component of the velocity into account, and calculates the suction effect of the propeller, and also the alteration in work done. Reference may be made and paper quoted. The results of his investigations show that there is a suction pressure, partly before and partly after the propeller, and this defect at the stern causes augmentation of resistance, which will be more fully described later.

§ **108. Actual Problem of the Screw Propeller.**—In the previous cases, the turning moment and thrust have been calculated by assuming certain motions to be impressed upon the water. Frictional resistance has been neglected, but it was taken into account by Professor Rankine in a theory already considered. In the absence of friction, the efficiency is $1 - s$, and therefore on a slip base is represented by a straight line, being unity when the slip is zero, and zero when the slip is unity. The general modification which results when friction is seen may be seen by the following argument. When the slip is zero, the direct thrust is

[1] *Transactions of the Institution of Naval Architects*, 1892.

zero, but the frictional resistance is not zero. The effect of driving the shaft is to cause a negative thrust. As the slip increases, the direct thrust increases, and, for a certain value of the slip, first balances the frictional effect. The net thrust is then zero, and so likewise is the efficiency. As the slip further increases, the forward thrust increases at a much greater rate than the negative thrust due to friction, so that the efficiency at first rapidly increases. For large values of the slip, the direct thrust is large compared with the negative thrust, and the efficiency approximated closely to the efficiency when friction is neglected. When the slip is unity the efficiency is zero.

Professor Rankine, in taking friction into account, assumed that the length of each element of blade projected longitudinally is constant, and that the velocity is represented by OF (Fig. 91). In that case

$$\text{velocity of gliding} = v_o\sqrt{1+q^2}$$
$$\text{length of element of blade} = l\sqrt{1+q^2}$$

If dF be the frictional force along the element, then

$$dF = fl\sqrt{1+q^2}\,.\,dr\,.\,v_o^2(1+q^2)$$
$$= \frac{f l\lambda v_o^2}{2\pi}(1+q^2)^{\frac{3}{2}}\,.\,dq$$

in which f is the coefficient of friction, and the notation is the same as in § 101. The longitudinal component

$$= dF\sin\theta = \frac{dF}{\sqrt{1+q^2}} = dT$$

the transverse component

$$= dF\cos\theta = dF\frac{q}{\sqrt{1+q^2}}$$

therefore the increment in couple

$$= \frac{dF\,.\,qr}{\sqrt{1+q^2}} = \frac{\lambda}{2\pi}\,.\,\frac{q^2 dF}{\sqrt{1+q^2}}\,.$$

Hence the diminution of thrust due to friction

$$= \frac{f l\lambda v_o^2}{2\pi}\int_o^q(1+q^2)dq = \frac{f l\lambda v_o^2}{2\pi}\left(q_1+\frac{q_1^3}{3}\right)$$

and the increment of couple

$$= \frac{\lambda^2}{4\pi^2} flv_o \int q^2(1 + q^2)dq$$

$$= \frac{\lambda^2}{4\pi} flv_o^2\left(\frac{q_1^3}{3} + \frac{q_1^5}{5}\right).$$

The diminution of work got is found by multiplying the first expression by $\lambda(1 - s)$, and of increment of couple by 2π.

Fig. 96 shows the curve of efficiency for the *Warrior*, in which the boss is taken into account. The pitch was 30 feet, the outer

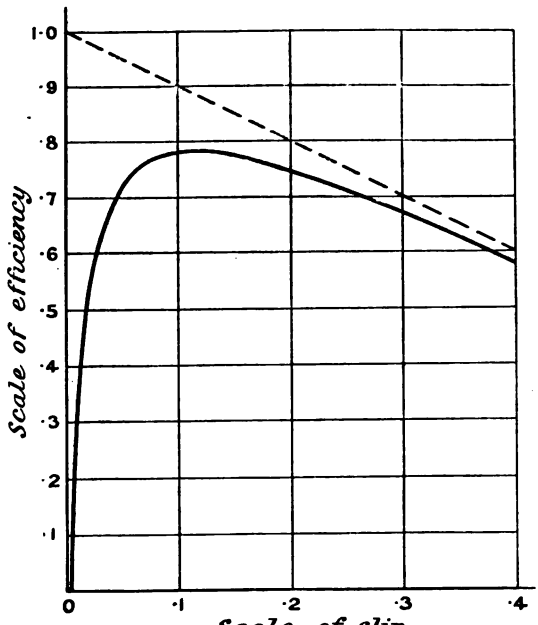

FIG. 96.

diameter 24 feet 6 inches, the inner diameter 6 feet, the length of blade (two blades) was 3 feet 6 inches × 2 = 7 feet, the revolutions per minute were 54·24, the speed 14·25 knots, and $f = 0·008$. It will be seen that the maximum efficiency is 0·78, corresponding to a slip of 10 per cent. At a slip of 20 per cent. the efficiency is 0·74; and at 30 per cent. the efficiency is 0·66.

Mr. W. Froude's theory.—Mr. W. Froude, instead of assuming the normal component of pressure determined from the subsequent motion, determined it experimentally.

In Fig. 97, OL is the circumferential velocity, LF the longitudinal velocity on the assumption that the screw works in a solid nut, LE the velocity of the ship, θ the pitch angle, and ϕ the *slip* angle. The velocity with the plate moves through the water is

$$OE = V \operatorname{cosec} (\theta - \phi)$$

and, using the formula for normal pressure (§ 49), the

$$\text{normal pressure} = P = aAV^2 \operatorname{cosec}^2 (\theta - \phi) \sin \phi$$

in which A = area of the element and a a constant, which, for small angles of ϕ, may be taken as 1·7.[1]

The estimation of the tangential force is a difficult matter, because of the indeterminateness in stating the velocity with

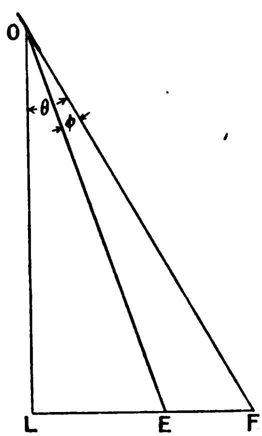

FIG. 97.

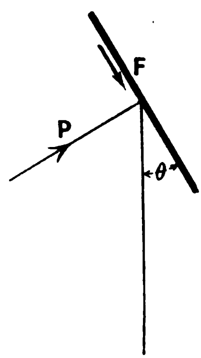

FIG. 98.

which the water glides over the plate. If there were no slip, the velocity of gliding would be represented by OF, the assumption made by Rankine. In an actual case, if the plate advance into still water, the streams would subdivide, as shown in

[1] This is for a flat plate. Professor Cotterill considered that a nearer approximation for a propeller blade is 2·5 (§ 116).

Fig. 50. The tangential force along the plate might therefore be reduced. But, as already pointed out in § 106, there is an eddying mass behind the plate, and the eddies due to the friction of the plate would cause an additional indirect resistance.

Thus, let (Fig. 98) frictional force

$$= F = fA \,.\, OE^2$$
$$= fAV^2 \operatorname{cosec}^2 (\theta - \phi).$$

Ordinarily for a varnished surface 2 feet long $n = 2$, and the resistance per square foot, at 10 feet per second, is 0·41 pound, so that $f =$ 0·0041. For both sides, f may be taken to be 0 008.

§ 109. **Estimation of Efficiency of Element.**—Thus, in Fig. 99,

thrust $= T = P \cos \theta - F \sin \theta$

transverse force $= M = P \sin \theta + F \cos \theta$

useful work done per second

$$= TV$$
$$= AV^3 \operatorname{cosec}^2 (\theta - \phi)\{a \sin \phi \cos \theta - f \sin \theta\}$$
$$= aAV^3 \operatorname{cosec}^2 (\theta - \phi)\{\sin \phi \cos \theta - k \sin \theta)\} \quad . \quad (7)$$

in which

$$k = \frac{f}{a} = \frac{0·008}{1·7} = \frac{1}{200}, \text{ say.}$$

Work put in

$$= M \times \text{rotary velocity}$$
$$= MV \cot (\theta - \phi)$$
$$= aAV^3 \operatorname{cosec}^2 (\theta - \phi) \cot (\theta - \phi)\{\sin \phi \sin \theta + k \cos \theta\} \quad (8)$$

Hence

$$\text{efficiency of element} = \frac{\sin \phi \cos \theta - k \sin \theta}{\cot (\theta - \phi)\{\sin \phi \sin \theta + k \cos \theta\}} \quad (9)$$

$$= \frac{\tan (\theta - \phi)}{\dfrac{\tan \theta + \dfrac{k}{\sin \phi}}{1 - k \dfrac{\tan \theta}{\sin \phi}}}$$

$$= \frac{\tan (\theta - \phi)}{\dfrac{\tan \theta + \tan \phi'}{1 - \tan \theta \tan \phi'}} \quad \text{in which } \tan \phi' = \frac{k}{\sin \phi}$$

$$= \frac{\tan (\theta - \phi)}{\tan (\theta + \phi')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

§ 110. **Conditions for Maximum Efficiency.**—In this expression θ and ϕ are independent, but ϕ and ϕ' are connected by the relation —for small values—$\phi\phi' = k$.

For a given value of ϕ (and therefore of ϕ') the efficiency is a maximum when—

$$\theta = 45^\circ + \tfrac{1}{2}(\phi - \phi')$$

and is then

$$\text{efficiency} \quad = \eta = \frac{\tan\left(\frac{\pi}{4} - \frac{\phi + \phi'}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\phi + \phi'}{2}\right)} = \left(\frac{1 - \tan\frac{\phi + \phi'}{2}}{1 + \tan\frac{\phi + \phi'}{2}}\right)^2$$

The maximum value of the efficiency will be the greatest possible when $\tan\frac{\phi + \phi'}{2}$ is a minimum, that is, when $(\phi + \phi')$ is a minimum subject to the condition $\phi\phi' = k$. This happens when $\phi = \phi'$, in which case $\theta = 45^\circ$.

The greatest maximum efficiency

$$= \left(\frac{1 - \tan\phi}{1 + \tan\phi}\right)^2 = \left(\frac{1 - \phi}{1 + \phi}\right)^2 = \left(\frac{1 - \sqrt{k}}{1 + \sqrt{k}}\right)^2.$$

Numerically, when these conditions hold—

$$\phi = \sqrt{k} = \sqrt{\frac{1}{200}} = \frac{1}{14{\cdot}14} = 0{\cdot}0708 \text{ radians} = 4{\cdot}05^\circ$$

$$\eta = \left(\frac{1 - 0{\cdot}0708}{1 + 0{\cdot}0708}\right)^2 = 0{\cdot}75.$$

Also

$$\begin{aligned} 1 - s &= \tan\,(45^\circ - 4{\cdot}08^\circ) \\ &= \tan 41^\circ = 0{\cdot}869 \\ s &= 0{\cdot}13. \end{aligned}$$

Thus, the greatest possible efficiency of an element is 75 per cent., corresponding to a pitch angle of 45°, and a slip of 13 per cent.

Again, the efficiency will be a maximum for a given value of θ, that is, for a given pitch ratio, when

$$\phi = \frac{k}{\phi'} = -\,2k\cot 2\theta + \sqrt{k^2 + 4k^2\cot^2 2\theta} \quad . \quad (11)$$

ϕ, ϕ' being assumed small. Having calculated ϕ and ϕ', then the maximum efficiency will be obtained from

$$\eta = \frac{\tan(\theta - \phi)}{\tan(\theta + \phi')} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This result is obtained by differentiating

$$\frac{1 - \tan\frac{\phi + \phi'}{2}}{1 + \tan\frac{\phi + \phi'}{2}}$$

with respect to ϕ, making use of the relation $\phi\phi' = k$.

§ 111. **Expressions for Thrust, Turning Moment, and Efficiency.**—In many cases it is convenient to express the preceding results in terms of the slip ratio s, and the quantity q (= cot θ), which is π times the diameter ratio, and varies from $\frac{1}{2}$ to 1 (that is, the ratio of diameter to pitch). Referring to Fig. 99,

$$\left.\begin{aligned}
\sin\theta &= \frac{1}{\sqrt{1 + q^2}} \\
\cos\theta &= \frac{q}{\sqrt{1 + q^2}} \\
\sin\phi &= \frac{sq}{\sqrt{1 + q^2}\sqrt{q^2 + (1 - s)^2}} \\
\tan\phi &= \frac{sq}{q^2 + (1 - s)} \\
\cot(\theta - \phi) &= \frac{q}{1 - s} \\
\operatorname{cosec}(\theta - \phi) &= \frac{\sqrt{q^2 + (1 - s)^2}}{1 - s} \\
V &= (1 - s)v_o.
\end{aligned}\right\} \quad (13)$$

The thrust in the elementary area dA will then be

$$dT = dA \, . \, v_o^2\left\{\frac{asq^2\sqrt{q^2 + (1 - s)^2}}{(1 + q^2)} - \frac{f(q^2 + 1 - s^2)}{\sqrt{1 + q^2}}\right\} \quad (14)$$

the useful work being

$$v_o(1 - s)d\mathrm{T}.$$

The work put in

$$= dA \,.\, v_o^3\left\{\frac{asq^2\sqrt{q^2 + (1 - s)^2}}{1 + q^2} + \frac{fq^2(q^2 + \overline{1 - s^2})}{\sqrt{1 + q^2}}\right\} \quad (15)$$

The efficiency of the element is—

$$\eta = (1 - s)\frac{sq^2 - k\sqrt{(1 + q^2)(q^2 + \overline{1 - s^2})}}{sq^2 + kq^2\sqrt{(1 + q^2)(q^2 + 1 - s^2)}} \quad . \quad . \ (16)$$

If friction be neglected, $k = 0$ and $\eta = (1 - s)$. This expression shows how the efficiency depends on the diameter ratio $\left(\frac{q}{\pi}\right)$ for a given slip, and on the slip for a given diameter ratio. It only refers to an element.

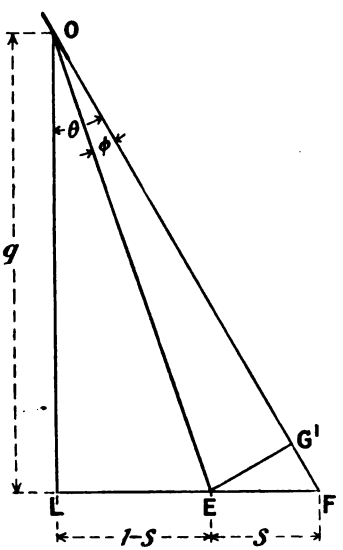

FIG. 99.

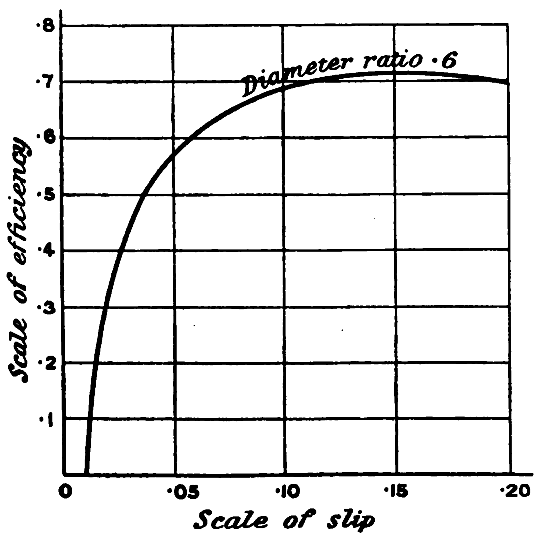

FIG. 100.

§ **112. Curves of Efficiency of a Slip Base.**—Fig. 100 shows the curve of efficiency on a slip ratio base for a diameter ratio of 0·6,

that is, for a value of $q = 1{\cdot}884$. The general nature of the curve has already been pointed out. In this case $\theta = 28°$, and therefore the value of ϕ for maximum efficiency (equation 11) is 0·0641 radians, or 3·68°. The value of ϕ' (taking $k = \frac{1}{200}$) is 0·078 radians, or 4·49°. The maximum efficiency, or

$$\frac{\tan\,(28° - 3{\cdot}68°)}{\tan\,(28° + 4{\cdot}49°)} = 0{\cdot}71.$$

The corresponding slip is 0·15 (Fig. 101). When the numerator in equation 16 is zero, q has some definite value, so that the curve of efficiency meets the vertical axis below the origin, showing a negative efficiency.

§ 113. Curves of Maximum Efficiency on a Diameter Ratio Base.—To show how the maximum efficiency depends on the diameter

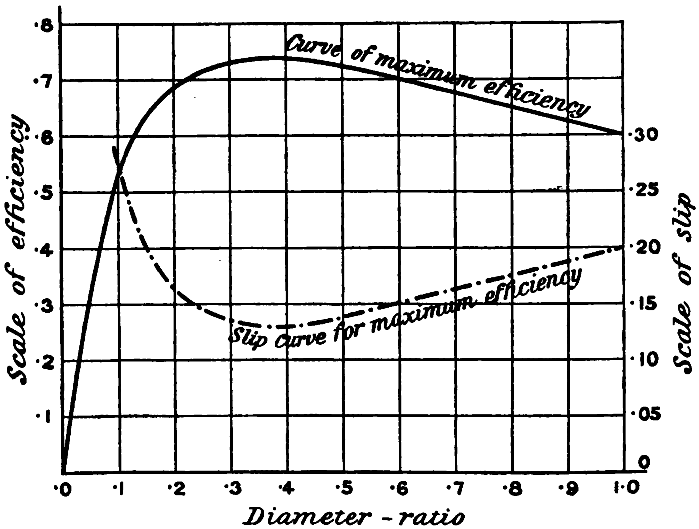

FIG. 101.

ratio. Fig. 101 shows this curve, and the curve of corresponding slip plotted on a diameter ratio. Knowing θ, and assuming $k = \frac{1}{200}$, ϕ is obtained from equation 11. The slip is then obtained from the fifth expression in equation 13, and the maximum efficiency obtained

from equation 12. It will be noticed that the curve of maximum efficiency rises rapidly, reaches a maximum of 0·73, when the slip is 0·13, and gradually decreases. Thus, if the diameter ratio be greater, a comparatively large increase in diameter will not affect the maximum efficiency. The slip rapidly decreases, reaches a minimum, and then gradually rises.

§ 114. **Comparison between Rankine's and Froude's Theories.**[1]—Rankine's treatment is the same as that of an oblique paddle. He assumes that the velocity which the water particles receive in a direction normal to the plate is the component velocity of the plate taken in the direction of the normal to the absolute. The absolute normal velocity of the particles is thus obtained. The quantity of water acted upon per second by an element of the plate is obtained on the assumption that the column is complete, that is, all the water passing through the space swept out by the elements of the blade is thrown back with the same velocity. The momentum, and therefore the thrust in a direction normal to the plate, is at once calculated, and the fore-and-aft component of this will give us the thrust, and the transverse component the transverse force to be applied.

In Froude's method, the normal pressure and tangential force, instead of being calculated, is obtained by experiment; otherwise, the methods are not dissimilar. In passing from a plate to a screw-blade, the following difficulties arise:—

(1) The change from motion in a straight line to motion in a curve.

(2) The change from a flat surface to a curved surface.

(3) The fact that the quantity of water acted upon is not unlimited as in a plane plate, but cannot exceed that which passes through the screw disc.

The effect of the first two is probably so slight as to be neglected; but the effect of the third might be very great. According to Froude's theory, if a remain constant, the thrust can be indefinitely increased by increasing the area of the blade. If the quantity of water acted upon is unlimited, the thrust will

[1] Professor Cotterill, F.R.S., *Transactions of the Institution of Naval Architects*, 1879.

undoubtedly increase continually with the area—as in a flat plate; but, in a screw propeller, the quantity of water acted upon is not unlimited, but is limited by the amount which can pass through the screw disc. Consider an element of the blade of length l and of small width. As the length increases, neglecting friction, the quantity of water and the thrust increase. There must be a limit when the thrust is sufficient to operate on the whole quantity of water passing through the screw, so that any additional increase in length will not operate on any additional water, and will, therefore, not further increase the thrust. The thrust may possibly be decreased on account of the friction of the increased blade area.

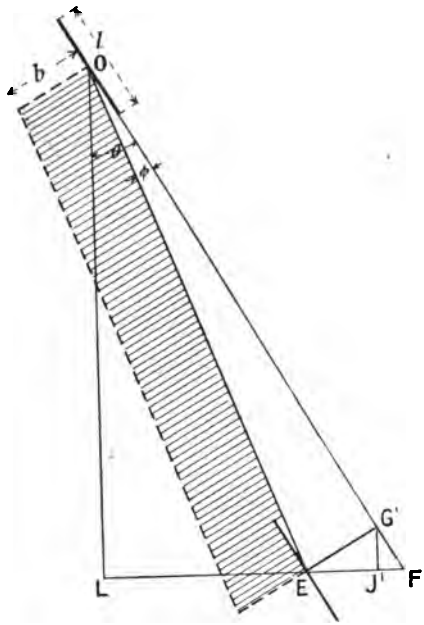

FIG. 102.

§ 115. Limiting Length of Blade.—To find a limiting length of blade on Froude's theory, in Fig. 102—

Let l = length of plate;
V = the velocity of plate through still water;
ϕ = the "slip" angle.

The normal pressure $= 1{\cdot}7lV^2 \sin \phi$.

Also, velocity of particles in contact with the plate in direction of normal to the plate

$$= V \sin \phi.$$

The plate, in one second, acts upon a certain quantity of water. Suppose that the water thus acted upon is supposed to have a

normal uniform velocity of V $\sin\phi$, and to extend a normal distance b, so that

$$b = \text{breadth (minimum) of uniform disturbance,}$$

then, if Q be the water operated on per second,

$$Q = \text{shaded area} = Vb\sin\phi = Vb$$

for small values of ϕ.

From the principle of momentum, normal pressure

$$= \frac{w}{g}QV\sin\phi = 2b\,V^2\sin\phi = 1{\cdot}7\,lV^2\sin\phi$$

by Froude's theory.

$$\therefore b = 0{\cdot}85l.$$

Thus the minimum breadth of water affected by the plates is $0{\cdot}85l$; and if, for any particular reason, the breadth of water striking the plate is less than this, the full pressure will not be developed. Such a case might happen if two parallel plates are moved through the water; it follows that if the perpendicular distance between the plates be not $> 0{\cdot}85l$, much of the water which would otherwise strike the rear plate will be intercepted by the front one, and the pressure will be less than that given by the formula.

To apply this to the screw, consider a development (as shown). In that case

$$b_o = a\sin\theta$$
$$b = l\cos\theta$$
$$l = b\sec\theta$$
$$\therefore a\sin\theta = 0{\cdot}85\,b\sec\theta$$

or

$$b = 1{\cdot}18\,a\sin\theta\cos\theta.$$

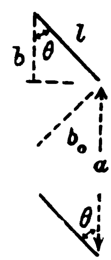

Fig. 103.

In this expression,

a = transverse arc between two blades ;

b = projection of length of blade on diametrical plane.

Generally, if

C = whole circumference ;

B = aggregate lengths of circular arcs formed by projecting on a diametrical plane.

So that

$$B = 1{\cdot}18\,C\sin\theta\cos\theta.$$

In order that the theory of a revolving plate may be applicable

without sensible deduction, the fraction of the circumference occupied by the blades must not be greater than $1{\cdot}18 \sin\theta \cos\theta$, or, more probably, considerably less. This is true whatever the number of blades, for the water is intercepted by the blade which lies immediately in front of it, or, if there be but one blade, by that blade itself in the course of the revolution immediately preceding.

If the blades be at a sufficient distance apart, clearly the whole water will not be affected, and Rankine's assumption that there is a sternward stream of uniform velocity will not be satisfied. Rankine's formula would, therefore, give too great a thrust. But if the row of plates are sufficiently near together, all the water will be sensibly affected in the same way, and will receive the same velocity, and consequently Rankine's assumption would be justified.

Thus the limiting case of Froude's theory is when the thrust given by that theory is equal to that given by Rankine's theory. Assuming the same velocity of gliding in the two cases, the frictional part of the thrust will be the same in each case, so that the thrust only may be considered. Thus to find the dimensions of the blade which give a complete column, the thrusts of each element must be the same. Considering each element to form its corresponding cylindrical column, every particle of which moves in its spiral path, and assuming that l represents the net width of all the blades at radius r, and taking unit depth (Fig. 102), thrust of element as given by Froude—

$$= 1{\cdot}7\, l^2 \,.\, \mathrm{OE}^2 \sin\phi \,.\, \cos\theta \quad (\S\ 99)$$

and by Rankine—

$$= \frac{w}{g}\, \mathrm{EG}' \cos\theta \,.\, 2\pi r \,.\, \mathrm{LJ}'$$

$$= 4\pi r \,.\, \mathrm{OE} \sin\phi \cos\theta \times \mathrm{OE}\,(\sin\overline{\theta - \phi} + \sin\phi\cos\theta) \quad \text{(Fig. 90)}$$
$$= 4\pi r \,.\, \mathrm{OE}^2 \sin\theta \cos\theta \sin\phi \cos\phi.$$

Whence, equating

$$l = 1{\cdot}18 \sin\theta \cos\phi \times 2\pi r.$$

Usually, $\cos\phi$ is practically unity for all ordinary slips, whence

$$l = 2\pi r \times 1{\cdot}18 \sin\theta, \text{ approximately.}$$

Since

$$\sin\theta = \frac{1}{\sqrt{1+q^2}}$$

$$r = \frac{3}{2\pi} q$$

$$l = 1{\cdot}18 \frac{\lambda q}{\sqrt{1+q^2}}.$$

Professor Cotterill came to the conclusion—from an examination of a number of cases—that Rankine's theory does not form a complete column, and that Froude's theory is the one to take in all cases of calculation.

§ 116. **Extension to the whole Blade.**—The above investigation, as well as the curves, refer to an element of the blade. In extending the theory to the whole blade, in addition to knowing the shape of the blade, a and f must be assumed to have the same value for each element. Integration will give the total thrust and total work put in, and so the efficiency of the whole blade is obtained.

Not infrequently the developed area of the blade is an ellipse, one extremity of the axis major being at the axis of the shaft, and the other at the tip. A certain part of the ellipse is cut away by the boss. Sometimes the curve is fuller near the tip than the ellipse would give. If the developed area be assumed a rectangle, the equations 14 and 15 in § 111, can be evaluated, and the total work put in and got out expressed in terms of l, λ, v_0, q, and s, and the coefficients a and f. It is not proposed to give the analysis, nor to draw curves, similar to Figs. 101, 102 for the whole screw.

The curves, however, for the whole screw will be similar in form to the curves drawn for an isolated element. The greatest maximum efficiency (§ 113) will not, however, take place when the pitch angle is 45°—that is, when the diameter ratio is 0·319—but at a greater diameter ratio. For, if the extreme diameter ratio be

0·319, the tips will be working at their greatest efficiency, but the body of the blade—particularly those parts for which the diameter ratio is less than 0·2—will be working at very much reduced efficiency, and the efficiency of the whole screw will probably be less. But if the extreme diameter ratio be fairly large, so likewise will no part (except that cut out by the boss, which is non-effective) have a small diameter ratio, and so the average efficiency will be fairly high. The value of that efficiency will not, probably, vary to any great extent, even for considerable variations in the extreme diameter ratio.

The values of a and f that have to be taken in the expression for the thrust and for work put in, in the case of the whole screw, are somewhat doubtful. For an isolated thin element moving in a straight line through undisturbed water of unlimited extent, the values may be taken to be 1·7 and 0·008 respectively. In considering a screw blade, the element is not isolated, and the section of the blade is plano-convex, being thicker the nearer the boss (Fig. 103). The effect of the section of the blade is as shown, and Professor Cotterill suggested that for 1·7 the coefficient 2·5 ought to be substituted. The coefficient f represents not only a surface effect, but the total free force in the direction of the plane of the driving force, so that probably f increases from the tip inwards. The form of the blade will not only modify f but also a. The effect of the section is to cause an excess pressure from A to B, say, and a defect pressure from B to C. Thus, due to the action of the curved back, there is a resultant force perpendicular to the face, and also along the face. As the slip increases, the suction part on ABC increases, so that probably a and f will both increase.

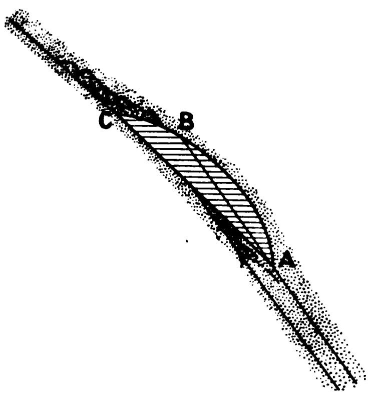

FIG. 104.

To remedy this defect, the section is, sometimes, as shown in Fig. 105, the leading edge at the back being tangential to the direction of motion. Thus, a and f will probably, in some way, depend on diameter-ratio.

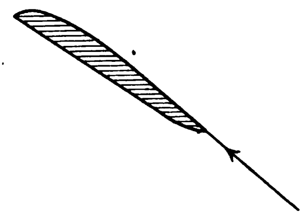

Fig. 105.

In attempting to estimate the thrust and turning moment for a screw of given shape, the logical method is to split the blade into a sufficient number of elements, and to find the thrust and turning moment for each element, so that, by addition, the total thrust and total work put in can be obtained. The work would probably be simplified by plotting curves of work got at and work put in, say, on a diameter-ratio or on a slip-ratio base. The ratio of the areas would give the efficiency of the screw. It is beyond the scope of this work, and such attempt to consider the whole blade would probably lead to fallacious results.

Whatever the merits or demerits of any particular theory, in order to judge of the performance of the whole screw recourse must be had to experiment. In making experiments on the performance of screws under different conditions, precautions must be taken so that the screw advances into undisturbed water. Experiments to determine the performances of screws cannot be made on screws which propel vessels, on account of extraneous disturbing factors entering into the problem. The screw must either project in front of a launch, as in the experiments of Messrs. Thorneycroft and Froude, or must be carried in a framework, as in Mr. R. E. Froude's experiments in the Admiralty tank. These experiments are, practically, the only available experiments from which the performance of ordinary-shaped screws may be inferred. The results are contained in three papers read before the Institution of Naval Architects in the years 1883, 1886, 1892.

§ 117. Dynamometrical Apparatus used by Mr. R. E. Froude.—In making experiments on screws, the quantities to be measured are—

(1) The thrust delivered by the screw.

(2) The turning movement delivered to the screw spindle.

(3) The linear speed of advance of the screw relative to still water.

(4) The revolutions of the screw shaft.

Fig. 106 shows the dynamometrical apparatus used by Mr. R. E. Froude at the Admiralty Works at Haslar. The screw on which experiments are made is mounted on the forward end of a shaft, A, 3 feet 6 inches long, the bearings of which are bracketed down from a frame above the water level, and which is driven with mitre gearing at its after end by a vertical spindle, B, the top bearing of which is above the water frame. This above water frame, E, is mounted on a delicate parallel motion, DD, constraining it

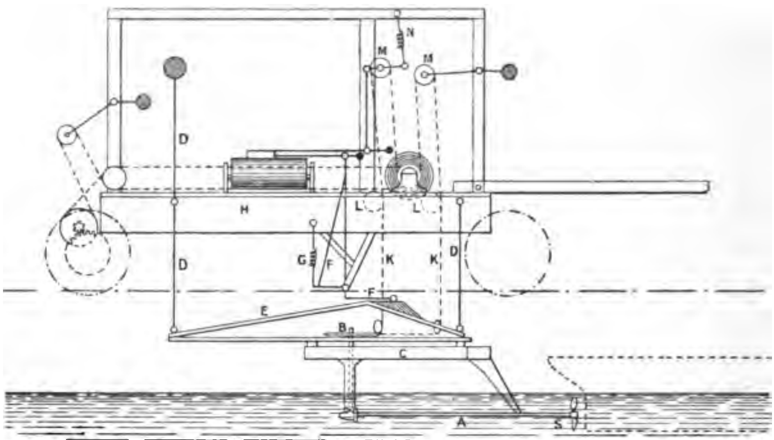

Fig. 106.

vertically and transversely, but leaving it perfectly free fore and aft; and its tendency to fore-and-aft motion, which consists of the forward thrust of the screw minus the resistance of the mechanism in the water, is measured automatically by means of a spiral spring, recording its extension on a revolving paper cylinder. The linkage is clearly shown by FF, and the spiral spring by G. The whole is mounted on a truck running on the straight and level railway, which extends throughout the length of the experimental tank a foot and a half above the water surface. The vertical spindle B, which drives the screw, is driven by cord belts, KK, and a system of poly-grooved pulleys, by the truck wheels, so that duly speeding

these pulleys any desired linear travel per revolution can be rigorously assigned. The cord is guided over guide pulleys, LL, on the frame, and, to avoid constraining the parallel motion, the centres of the pulleys should be in the line of the fulcra of the parallel motion. Any desired linear speed can be assigned to the truck by the governor of the engine which drives it. The final cord belt, which drives the spindle, passes over a system of delicate levers and pulleys, MM, by which the difference in the tension of the two parts of the belt, which is the measure of the turning movement applied to the mechanism, is automatically recorded on the same cylinder as the fore-and-aft force of the frame. In Fig. 107 the spring which measures the turning is denoted by N.

When twin screws are used, each screw has its own frame, the two frames being mounted, at their proper distance apart, on the same parallel motion, the driving-belt passing successively over the sheaves on the two vertical spindles, so that the diagram in that case records the sum of the fore-and-aft forces delivered by, and the sum of the two turning movements applied to, the mechanisms of the two screws.

For eliminating the resistance of the frame, etc., to its passage through the water, from the fore-and-aft force, and the friction of the bearing, etc., of the mechanism so as to convert these measures into true thrust delivered by, and true turning moment applied to, the screw, various expedients were adopted. Mr. Froude suggests that there might be slight inaccuracy in both these directions; he considers that the measure of the thrust is the most trustworthy, and may be considered very accurate; he also considers that comparative turning-moment results of experiments included within a short interval of time, may be trusted with more confidence than their absolute results.

In the experiments the truck carrying the apparatus above described is joined to the somewhat similar truck (Fig. 107) running on the same railway, the models being for this purpose attached beneath it. The model M M is attached by a spring to a lever, B, turning about a fixed centre on the truck H H, which is of special construction. Thus the resistance is measured by the lever D; and the cylinder E is driven by gearing similar to that

shown in Fig. 105. When the two trucks are joined (compare Figs. 106, 107) the model may either be attached in its place or omitted, and the screw experiments made either behind the model or in undisturbed water, as desired; or, again, either the screw may be removed from the shaft, or the trucks disconnected so that the model also can be tried either alone or with the screw working behind.

The apparatus that carries the screw measures the thrust, speed, turning moment, and revolutions per minute; consequently the experiments on the screw working without the model alone measure the useful work to be done. The comparison

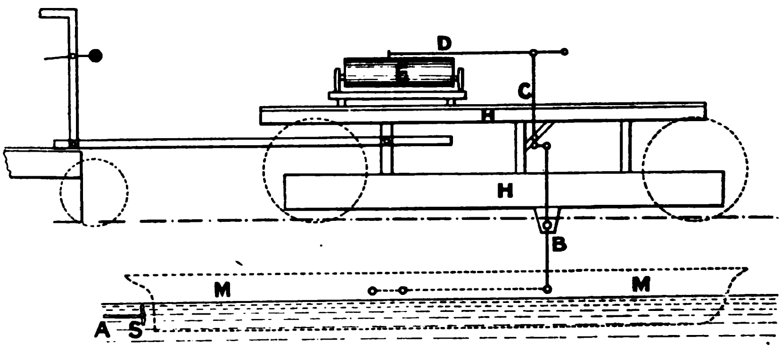

Fig. 106.

of these records with those obtained with the model or screw in continuation, shows the modifications introduced into the results by bringing the model and screw into conjunction, and these modifications constitute hull efficiency.

§ 118. **Interaction between Model and Screw.**—In extending the results deduced from experiments on model screws advancing into still water to the case where the screw works behind and propels the model, various considerations present themselves. It has already been pointed out that the action of the screw in some way increases the resistance of the ship, so that the thrust delivered by the screw-shaft is greater than the towrope resistance of the ship at the same speed. The presence of the screw, therefore, affects the resistance of the ship. On the other hand, the

ship causes a following wake, which affects the performance of the screw. Thus the performance of each affects the other, and it is important to understand to what extent each is affected. This has been carried out by Mr. R. E. Froude[1] by the apparatus previously described.

The truck, carrying the model of the ship, with its automatic registering gearing, could be attached to the screw-truck, and so the performance of the screw obtained, at a given linear speed, with the model either in front or removed. He was able to measure—

I. (α) The resistance of the model alone.
   (β) The resistance of the model with a screw behind it.

II. (α) The thrust and turning moment of a screw with no model.
   (β) The thrust and turning moment of a screw with the model in front of it.

The effect, therefore, of the presence of each on the performance of the other could be obtained. When twin screws are used, each screw has its own frame, the two screws being mounted, at the proper distance apart, on the same parallel motion, the driving belt passing successively over the sheaves on the two vertical spindles, so that the diagram in that case records the sum of the net fore-and-aft forces delivered, and the sum of the turning moments applied to the mechanism of the two screws.

The precise constitution of the wake is extremely complicated. In attempting to calculate the thrust of the screw, it is necessary to know, approximately, the variation of speed in the wake, because the thrust depends on the *real* slip, and therefore on the linear speed of advance of the screw relative to the water in which it works. Now, the forward velocity at any point is due to three causes: (1) friction; (2) stream-line motion; (3) wave motion. The effect of the first two is always to cause a forward motion of the water, and therefore to make real slip, on which the thrust depends, greater than the apparent slip. The effect of the third is to cause a forward or backward motion according as there is a crest or hollow. Expressed generally, the maximum forward velocity is likely to occur in particles which flow past the fuller portions of the stern and near the water-line; and the

[1] *Transactions of the Institution of Naval Architects*, 1883.

velocity will be less sideways away from the ship, or downwards, the run being finer.

The only experimental data are given by Mr. Calvert,[1] who experimented on a boat $28\frac{1}{2}$ feet long, $3\frac{2}{3}$ feet beam, $1\frac{1}{2}$ feet draught, 3 tons displacement, and 1500 square feet of wetted surface. A wire frame was built over the stern and under the stern, which carried small tubes, the frame being shifted in different positions; the tubes held to gauges on a common stand. By sucking each tube, the water in the tubes stood, when at rest, at the level of the water outside. When the boat moved, the water level was raised to an extent depending on the velocity of the water. By carrying a similar tube on an outrigger, the height was obtained for a tube going in still water, and taking this as datum, the percentage variation of the different points could be determined. As the object of such experiments was to determine the velocity at different points, certain corrections had to be applied to the observed readings. The height in the tube gives the velocity head and pressure head. The latter depended on the height of the stern wave. The depression of the boat was also obtained. He analyzes the results and tabulates them.

§ 119. **Consideration of whole Problem of Interaction of Screw and Hull.**—The primary effect of the hull is to cause a wake which, for a given revolution, causes the real slip and thrust to be greater than it otherwise would be. The primary effect of the screw is to cause an augmentation of resistance. This augmentation might be due to three causes, namely—

(1) Effect of centrifugal action.

(2) Acceleration of water in advance of screw.

(3) Ordinary suction.

The effect of centrifugal action in causing a reduction of thrust has already been discussed (§§ 102, 104). The reduction of pressure varies, in a given screw, as the square of the slip; and, for a given slip and diameter, at first increases as the diameter ratio increases, reaches a maximum for a diameter ratio of about $\frac{1}{2}$, and then decreases again. This reduction of pressure in the rear of the screw will suck water through the screw, and so accelerates

[1] For full reference, see *Transactions of the Institution of Naval Architects*, 1893.

the water some distance ahead of the screw; and this acceleration will not be confined to water at the depth of the screw, but must extend to the surface, the level of which will be lowered. Ordinarily, on account of stream-line action, there would be an excess pressure at the stern, but owing to this sucking action this excess of pressure will not be fully developed. These effects at ordinary slip ratios, and in the case of a screw at the stern of a vessel with fine run, are probably very small. But if the access of water to the screw is impeded in any way, so that the effort of the screw to draw more water through it than naturally passes through by the motion of the vessel is hindered in any way, the level of the water between the screw and the stern of the ship is lowered to a much greater extent than before by the same speed of rotation of the race, and the stern of the vessel sinks further, thus causing a greatly increased augmentation of resistance. Thus the rotation of the race would not only modify the thrust, but it would modify the resistance.

The augmentation might likewise be due to acceleration in front of the propeller, according to Mr. R. E. Froude's theory, namely, that, of the total thrust and corresponding momentum generated, a portion must generally consist in a region of defect of pressure in front and an excess pressure behind. The defect of pressure in front would extend some distance ahead of the propeller, and would consequently cause a reduced pressure on the hull, as before.

Or again, augmentation might be due to the low pressure existing at the back of the blade, which will cause water to be sucked through the interval between the blades. This would cause the augmentation of resistance to be proportionately larger at small than at large slips; whereas, at small slips, centrifugal action would cause reduction to be very slight.

Another view of augmentation is to regard the stern of the vessel as partially protected by the screw blades from the inflow of water, in the same manner as the blades protect each other (§ 115).

However produced, the low pressure existing between the stern of the vessel and the screw must not only increase the resistance,

but, other things being the same, it must increase the thrust. The still-water chart includes both the effect of reduction of pressure in front as well as that behind the blade; but if the hull of the vessel impedes the free flow to the screw, the reduction of pressure in front of the screw will be greater than these curves include.

§ 120. Thrust Deduction Factor and Wake Factor.—Let ρ be the towrope strain at speed V_s, that is, the resistance of the model without the screw, V_s being the velocity of the vessel, T the total actual thrust experienced (as registered) when the screw works behind the model at stated number of revolutions, which, when the screw drives the model, must be equal to the total resistance of the model; ρ' the resistance when the screw is behind the model. The curves of ρ, ρ', and T may be plotted on a revolution base for the linear speed V_s. They will be somewhat as sketched in Fig. 108.

The curve of ρ will be a horizontal straight line; that of ρ' is very nearly straight, but slightly concave upwards; whilst the curve of T is as shown. Both the augmentation of resistance $(\rho' - \rho)$ and the actual thrust (T) increase with revolutions; that is, with slip—as previous considerations have shown (§§ 101, 102, 106). Mr. R. E. Froude considers that $\rho' - \rho$ may therefore be looked upon as a function of the thrust, and, instead of considering it as an augmentation of resistance, it may be taken as a reduction of thrust. The net thrust, or the effective thrust overcome, may then be said to be

$$T - (\rho' - \rho)$$

and the ratio of the net to the total is

$$\frac{T - (\rho' - \rho)}{T}$$

called the *thrust deduction factor*. When the screw drives the model $\rho' = T$, and this expression becomes

$$\frac{\rho}{T}$$

Fig. 108 shows the curves of ρ', ρ, and T. The thrust deduction factor is shown dotted. Within ordinary limits of revolution—

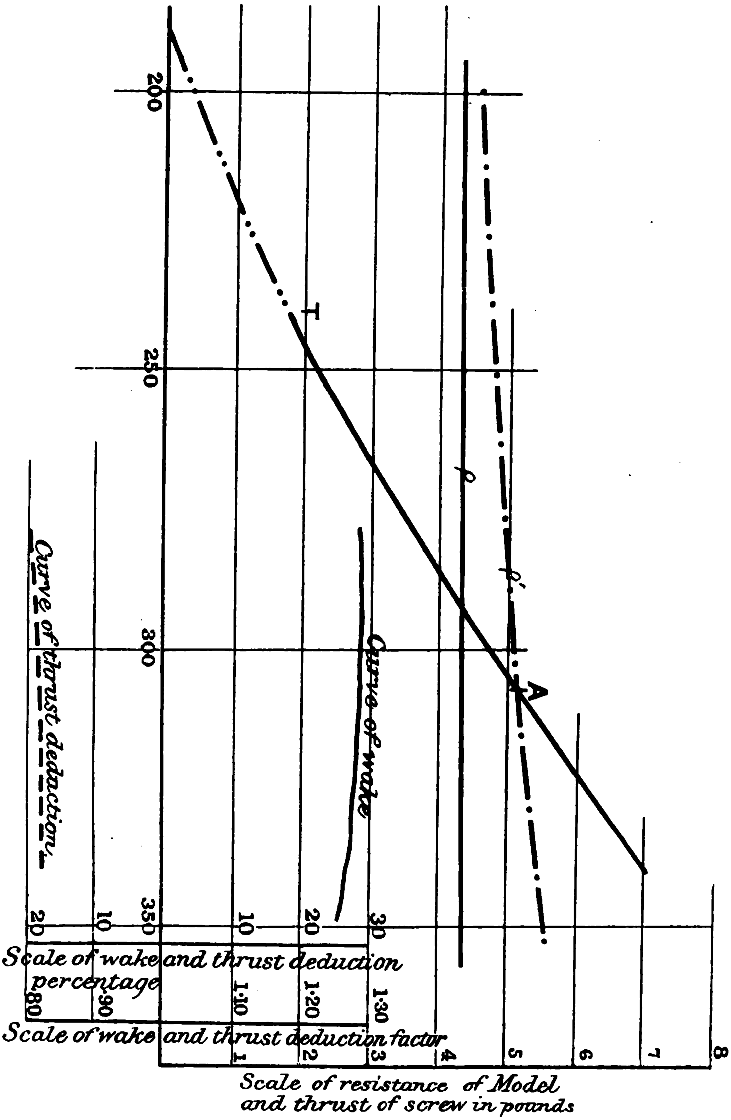
T
A
Curve of thrust deduction
Curve of wake
200
250
300
350
Scale of wake and thrust deduction percentage
Scale of wake and thrust deduction factor
Scale of resistance of Model and thrust of screw in pounds

FIG. 108.

that is, slip—the value is fairly constant; but it rapidly decreases, becoming zero when $\rho' - \rho = T$, and $-\infty$ when the actual thrust is zero. This latter is due to the fact that when the actual thrust is zero, there is still some augment of resistance. Mr. Froude suggests that this is probably due to the fact that the real slip will be positive in some parts (at the top) and negative at others (at the bottom), so that there might be an augmentation at the top and a reduction of resistance at the bottom. The suction at the top would have more effect on the hull than the excess pressure at the bottom—where the run is finer—and therefore there would be an increased resistance. But it may also be due to the secondary action previously pointed out (§ 119), namely, that when the actual thrust is zero, there is a reduced pressure at the back of the blade which might produce a sensible augmentation of resistance.

In Fig. 108 the wake factor $\left(\frac{V_s}{V}\right)$ curve is plotted. Neglecting any secondary effect, this ought to be constant, but it decreases slightly as the revolutions, and therefore the slip, increases. Since the wake is the production of the model and not of the screw, it ought to be constant at all slips. The variation might possibly be due to the want of uniformity in the wake, or to the secondary action just referred to.

§ 121. **Analysis of Work from Engines to Screw.** — Fig. 108 refers to the case where the screw does not propel the model. If this condition be satisfied, the state of affairs is represented by the point marked A, which is the point of intersection of the ρ' and T curves. At higher revolutions the case corresponds to where the model propels itself against a head wind, or is towing a model; at less revolutions, that of the model under sail. The curves, therefore, in the neighbourhood of A are apparently the only important parts. Assuming that the screw is propelling the model at speed V_s, then the work given out by the screw, if there were no interaction, would be ρV_s, the thrust of the screw being ρ. Actually the resistance is $\rho' = T$, and, therefore, the work that the screw apparently gives out is TV_s; but part of this is not due to the engines, but is the gain due to working in a forward wake. The useful work due to the engines is only TV, as it would be if the

screw advanced into still water at linear speed V, so that the gain of work due to working in a forward wake is $T(V_s - V)$. The whole transformation, from engine and screw, perhaps, may be best shown by the following classification :—

Work per second of engines in uniform motion.

Loss of work per second in mechanical friction.

Work per second delivered to the propeller by the engine shaft.

Available work per second from wake $T(V_s - V)$.

Available work due to engines per second (TV).

Loss of work in shock, friction, and causing useless motions.

Total available work per second (TV_s).

Work necessary to overcome augmented resistance per second $= V_s(T - \rho)$.

Towrope work per second. ρV_s.

Work per second still available in wake.

Work per second lost beyond recall.

The ratio $\frac{\rho V_s}{TV}$ is called the "hull efficiency."

Making the usual assumption that the turning moment on the shaft at given revolutions, with a model speed of V_s, is the same as that when the screw advances into still water at speed V, then if M be the turning moment, the

$$\text{net efficiency} = \frac{\rho V_s}{2\pi RM}$$

$$= \frac{\rho V_s}{TV} \times \frac{TV}{2\pi RM}$$

= hull efficiency × true efficiency of the screw at revolutions R, advancing into still water at speed V, determined from the assumption of a uniform work.

The ratio $\frac{TV}{2\pi RM}$ is the ultimate test of the performance of the propeller considered simply as a propeller; the ratio $\frac{\rho V_s}{2\pi RM}$ is the test of the value of the propeller as a means of actually propelling the ship. If the ship produced no wake, and the propeller no augmentation of resistance, these would be equal. The relation between them, therefore, involves the mutual interaction of ship and propeller, and is called the hull efficiency.

§ 122. **Values of Different Factors.**—The assumption of the same turning moment is not strictly fulfilled for reasons either due to the non-uniformity of the wake, or to secondary actions. The result can be expressed to take this difference into account:

Let M = turning moment at revolutions R, and linear speed V in still water;

M_s = actual turning moment at revolutions R and speed of model V_s, with screw behind model.

Then

$$\text{net total efficiency} = \frac{\rho V_s}{2\pi RM_s}$$

$$= \frac{\rho}{T} \cdot \frac{V_s}{V} \cdot \frac{M}{M_s} \cdot \frac{TV}{2\pi RM}$$

= thrust deduction factor × wake factor × rotary efficiency × true efficiency in still water at revolutions R and linear speed V.

Mr. R. E. Froude[1] made experiments on the effect of inward and outward turning on these three factors. In outward turning the starboard screw is right-handed, that is, turns in the direction of the hands of a watch when viewed aft and is driving the ship ahead, and the port screw is left-handed.[2] Experiment showed that, with the models tried, inward turning was favourable to efficiency than otherwise, the difference being almost within the limits

[1] *Transactions of the Institution of Naval Architects*, 1898.

[2] By inward turning floating objects are prevented from becoming jammed between the upper blade and ship's counter. Sometimes, however, the engine-room can be more conveniently arranged if the screws turn inwards.

of experimental error. The results show that as regards thrust deduction, outward turning has very slightly an advantage over inward; as regards hull efficiency $\left(\frac{\rho V_s}{TV}\right)$, inward has an advantage over outward; as regards rotative efficiency $\left(\frac{M}{M_s}\right)$, inward has an advantage over outward, the ratio being generally > 1 in the former, and less than one in the latter. The net result appears to be that, taking all the different factors into account, the inward turning has the slight advantage, but the results hardly do more than afford evidence that twin screws may be made to turn inwards instead of outwards without sacrificing efficiency. Taking the mean of both inward and outward turning, the following are the extreme values for different kinds of twin-screw ships:—

	Thrust deduction $\frac{\rho}{T}$	Wake factor $\frac{V_s}{V}$	Rotary efficiency $\frac{M}{M_s}$	Hull efficiency $\frac{\rho V_s}{TV}$	Total efficiency $\frac{\rho V_s M}{TVM_s}$
Battleships .	0·90 – 0·82	1·01 – 1·17	1·00 – 1·00	0·99 – 0·96	0·99 – 0·96
Cruisers . .	0·93 – 0·88	1·04 – 1·10	1·00 – 1·02	0·99 – 0·95	1·0 – 0·96
Destroyers .	0·99 – 0·96	0·99 – 1·04	0·99 – 1·02	1·00 – 0·98	0·99 – 1·0

In an extreme shallow-draught vessel with a Thorneycroft stern

$$\frac{\rho}{T} = 0{\cdot}88\,; \quad \frac{V_s}{V} = 1{\cdot}16\,; \quad \frac{\rho V_s}{TV} = 1{\cdot}02\,; \quad \frac{M}{M_s} = 1. \quad \therefore \text{ total } 1{\cdot}02.$$

§ **123. General Formula of Comparison in Screws.**[1]—The screws experimented upon were all of 8-inch diameter, and of pitch ratios 1·225, 1·4, 1·8, and 2·2 (that is, diameter ratios of 0·82, 0·72, 0·56, and 0·46). The developed blade was an ellipse, having one extremity of the major axis at the screw centre and the other at the tip, the minor axis being 0·4 times the major axis, and the boss diameter was 0·2 to 0·3 times the major axis. Two-, three-, and four-bladed propellers were tried.

This represents the Admiralty standard blade. It is often

[1] Mr. R. E. Froude, *Transactions of the Institution of Naval Architects*, 1883.

found, however, that, owing to the diameter being limited, sufficient blade area is not obtainable by these proportions, and in that case the elliptical form is still adhered to with an increased minor axis of 0·5 to 0·55 the major. If sufficient area cannot thus be obtained, even with four blades, as in the case of shallow-draught vessels, where the diameter is very limited, the elliptical form may be greatly departed from and the blade widened at the top. The boss diameter is 0·2 to 0·3 times the tip diameter.[1]

By the apparatus described, the thrust and efficiency of a given screw when run at a given linear speed and revolutions may be obtained. By varying the revolutions—keeping the linear speed the same—the thrust and efficiency of a given screw at a given linear speed is obtained for different slip ratios, so that the curves of efficiency and thrust can be plotted on a slip ratio base. The form of the curves are shown in Fig. 109.

The experiments appeared to show that at zero slip both the thrust and efficiency were zero, as it would be in a screw for which the frictional and other resistances are zero. In a thin plate, at zero slip, there would be a negative force and, therefore, a negative slip (compare § 116). The observed results would appear that when the nominal slip is zero the "effective" slip is great enough to cause a direct thrust sufficient to balance the fore and aft resistances.

Let T, V, D be the thrust, linear speed of advance and diameter corresponding to the slip *s*. Consider two screws, similar in all respects, of given extreme pitch ratio and working at the same slip, but at different linear speeds. Then all the other speeds are changed in the same proportion, and since all the forces are proportional to the product of the square of the linear dimensions and velocity, it follows that, in cases of geometrical similarity,

$$T \propto D^2V^2$$

$$\text{or } \frac{T}{D^2V^2} = \text{constant}$$

and the efficiency at a given slip is the same for all sized screws. Thus one curve plotted on a slip base may be taken to represent

[1] Barnaby's "Screw Propeller," p. 78.

the performance of all sized screws, at all linear speeds, which have the same extreme pitch ratio and the same shape, provided the curve, instead of representing a simple thrust curve at speed V and diameter D, has ordinates proportional to

$$\frac{T}{D^2V^2}.$$

The efficiency curve will, of course, be unaltered. Thus curves may be drawn giving the thrust of screws of different diameter,

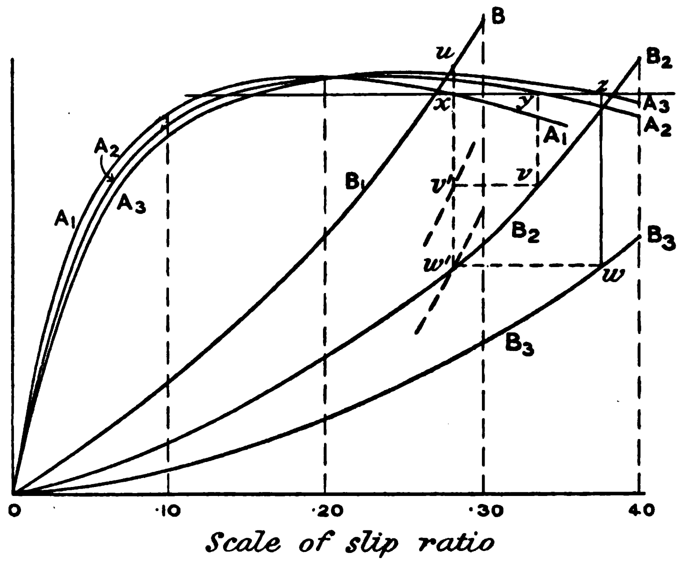

Fig. 109.

at different linear speeds, of given pitch ratio and slip. Suppose that a number of such curves are plotted on a slip base for different pitch ratios, and that a chart (Fig. 109) is obtained, which represents the performance of a screw of given shape.

The characteristics of these curves appear to be that, by suitably altering the horizontal scales, the curves of efficiency may be represented by one and the same curve. Thus, take a horizontal line, *xyz*, cutting the curves of efficiency A_1, A_2, A_3;

project the points x, y, z to intersect the curves B_1, B_2, B_3 in u, v, w, which give the thrust and slip at which screws of different pitch ratios have the same efficiency, the curve efficiency A being used as base. Now, suppose the horizontal scale is so altered that y and z are made to coincide with x; then v and w come to v' and w' respectively. If this "shift" be done for a sufficient number of points, there will be one curve of efficiency, and a number of modified thrust curves. The new horizontal scale is called the

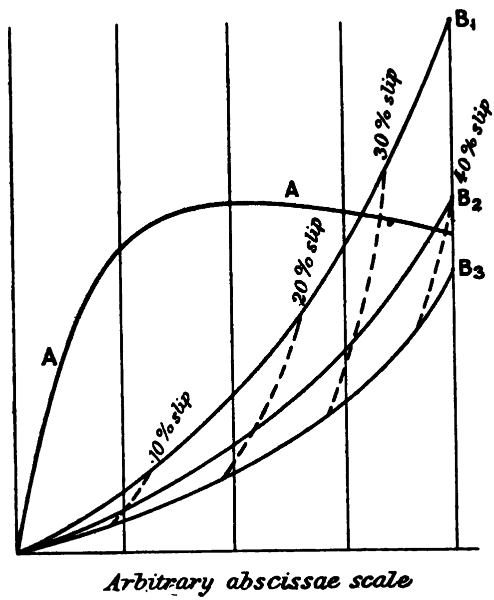

FIG. 110.

"abscissæ scale," and is clearly the true scale of slip for some particular pitch ratio. Except at this particular pitch ratio, horizontal distances will not give the slip, but the slip may be represented by a number of curves such as shown by the dotted curves in Fig. 110. Before distortion these dotted curves are vertical lines. It follows, from this construction, that different pitch ratios have the same maximum efficiency, but at different slips. Mr. Froude's experiments appear to show that this is true

within the range of pitch ratios ranging from 1·2 to 2·4. Mr. Barnaby considers that it is equally true for pitch ratios of from 0·8 to 2·5.

§ 124. **Solution of Screw Propeller Problem.**—Usually, the three quantities, T, V, and R (the revolutions) are known. In Fig. 111 choose any point, such as *a*, for which the pitch ratio, slip, and the value of $\frac{T}{V^2D^2}$ are known. Knowing T and V, the value of D is determined, and from the known pitch ratio the pitch P is found. But

$$s = 1 - \frac{V}{RP}$$

whence, knowing *s*, the revolutions may be determined. Thus, if the revolutions are arbitrary, there are a number of screws which will give the required thrust at the linear speed V, and which will have maximum efficiency; but, if the revolutions are fixed beforehand, obviously some other relation must be found. At any point *a*

$$\frac{V}{RP} = 1 - s.$$

Thus the value of $\frac{V}{RP}$, and, therefore, of $\frac{RD}{V}$, is determined. Corresponding to the point *a* in the pitch ratio curve 1·2, there will be the point *b*, so that the complete curve of a pitch ratio 1·2 may be plotted by taking a sufficient number of points. Thus a chart, such as that given in Fig. 112, gives all the solution necessary for an actual case.

Knowing T, V, and R, choose a point *a* on some particular ratio curve, whence D is determined. Project up to *b*, on the same pitch ratio curve, and so deduce a second value of D. By repeated trial, the two sets of curves will give the same diameter, whence—knowing the pitch ratio—the pitch can be found. From these curves, the dimensions of a screw of given pattern may be found for maximum efficiency; but they also show the effect on

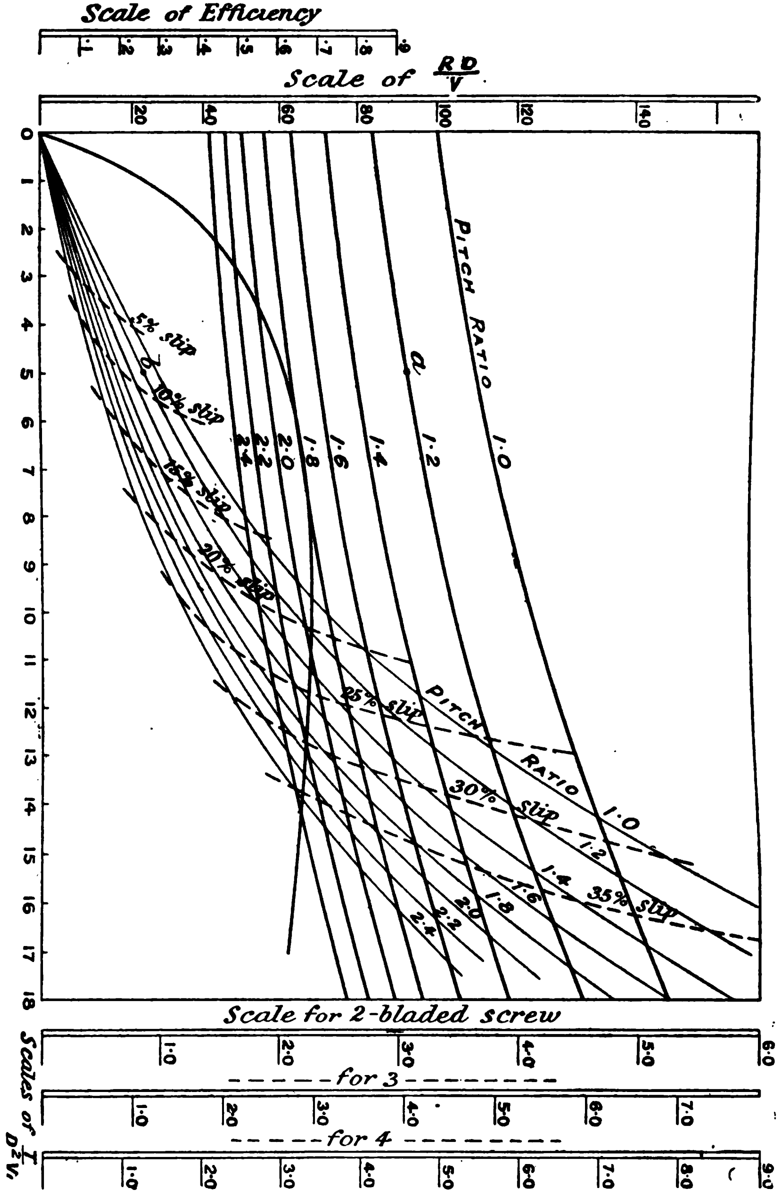
Scale of Efficiency
Scale of $\frac{RD}{V}$
Pitch Ratio
Pitch Ratio
5% slip
10% slip
15% slip
20% slip
25% slip
30% slip
35% slip
Scale for 2-bladed screw
for 3
for 4
Scales of $\frac{T}{D^2 V_r}$

FIG. 111.

the alteration in efficiency when these dimensions are modified to suit practical requirements.

The above refer to the case of a screw advancing into still water at the linear speed V. The thrust depends on the number of blades. Mr. R. E. Froude found that the thrusts of a four-, three-, and a two-bladed propeller (elliptical blade) were in the ratio

$$1 : 0{\cdot}865 : 0{\cdot}65 = 1\tfrac{1}{2} : 1\tfrac{1}{3} : 1$$

instead of

$$1 : 0{\cdot}75 : 0{\cdot}5 \text{ or } = 2 : 1{\cdot}5 : 1$$

if the thrust were increased in proportion to the number of blades. Thus, in Fig. 112, for a four-bladed propeller, the ordinates must be multiplied by 0·865 or 0·65 to obtain the results for a three- or a two- bladed propeller respectively; or, if they refer to a two-bladed propeller, the multiplier is $1\frac{1}{3}$ or $1\frac{1}{2}$ respectively. The scales are attached to Fig. 111.

In the actual case, the screw works behind the stern of the vessel. The motion of the vessel gives rise to a forward wake at the stern, so that the screw works in water which has a certain forward velocity impressed on it by the ship. The forward velocity will, no doubt, vary from point to point, but it is usually assumed that, so far as the propulsive action of the screw is concerned, the wake may be assumed to move forward with a uniform velocity. To prevent confusion, let V now represent the forward velocity of the screw relative to the water in which it works, and let V_s be the velocity of the ship, so that $V_s - V$ represents the absolute forward velocity of the wake. Then

$$V_s = wV$$

where w is a multiplier, greater than unity, which can only be determined by experiment. Again, since the speed of advance of the screw through the wake is V, it is exactly similarly circumstanced as if it advanced in undisturbed water at speed V, so that, T being the thrust delivered, the available work got out of the screw, which is due to the engines, is TV. Now, if ρ be the

towrope resistance—which, as already pointed out, need not be the same as T—then the effective or towrope work is ρV_s. It will be shown later that experiment shows that

$$\rho V_s = TV$$

nearly, so that the effective horse pressure

$$E.H.P. \propto TV \propto \frac{TV_s}{w}$$

and

$$\text{since E.H.P.} = p \,.\, \text{I.H.P.}$$

where p is the propulsive coefficient, so that

$$E.H.P. = p \,.\, I.H.P.$$

therefore

$$T \propto pw \frac{1}{V_s}.$$

The curves in Fig. 112 may refer to a particular screw, and represent all screws, the thrust ordinates being proportional to $\frac{1}{V^2D^2}$.

§ **125. Admiralty Method of designing Screws.**—For a given abscissa value—that is, for a given efficiency,

$$\frac{1}{V^2D^2} \text{ and } \frac{RD}{V}$$

are known corresponding to any particular pitch ratio. Let the former be denoted by t and the latter by r. Then—

$$t = \frac{1}{V^2D^2} \propto pw^3 \frac{1}{V_s{}^3D^2}$$

$$\text{whence} \quad D \,.\, \frac{V_s{}^{\frac{3}{2}}}{I^{\frac{1}{2}}} \propto \sqrt{\frac{pw^3}{t}} = O_D,$$

$$\text{and} \quad r = \frac{RD}{V} = w\frac{D}{V_s}R \propto \sqrt{\frac{pw^5}{t}} \,.\, \frac{RI^{\frac{1}{2}}}{V_s{}^{2\cdot5}}$$

$$\text{or} \quad R \,.\, \frac{I^{\frac{1}{2}}}{V_s{}^{2\cdot5}} \propto r\sqrt{\frac{t}{pw^5}} = O_R, \text{ say.}$$

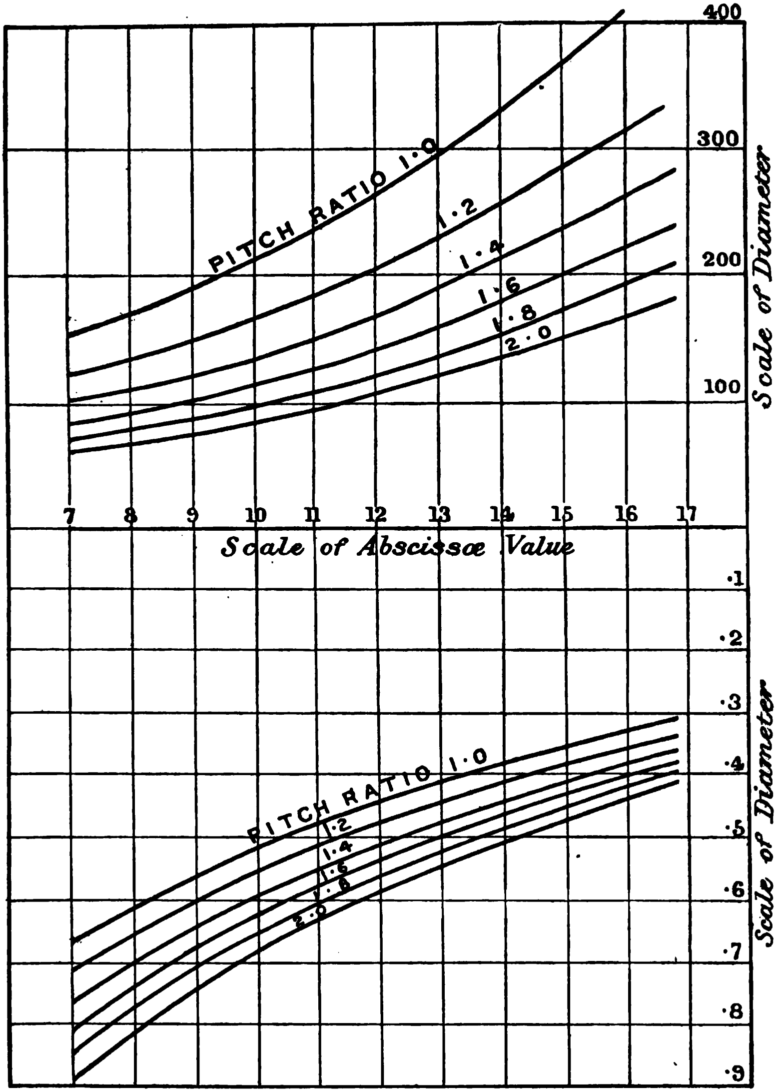
400
300
200
100
PITCH RATIO 1·0
1·2
1·4
1·6
1·8
2·0
Scale of Diameter
7
8
9
10
11
12
13
14
15
16
17
Scale of Abscissæ Value
·1
·2
·3
·4
·5
·6
·7
·8
·9
PITCH RATIO 1·0
1·2
1·4
1·6
1·8
2·0
Scale of Diameter

FIG. 112.

Knowing t and r, and assuming values for the propulsive coefficient and wake factor, O_D and O_R can be determined. The value of the propulsive coefficient varies little, and, for ordinary types of ships, may be taken as 0·5. The value of w will be discussed later, but an average value is $\frac{10}{9}$. If these coefficients are different, the necessary correction may be made.

The values of O_D and O_R are given in a tabulated form in Mr. R. E. Froude's paper, published in the *Transactions* for 1886. In Mr. Froude's paper, in 1892, the curves of O_R and O_D are plotted on an abscissæ value base. These curves are reproduced on square paper in Fig. 112. In the tables and curves—

R = tens of revolutions per minute
I = I.H.P. (for one engine in case of twin screws)
P and D = diameter of screw in feet
V_s = speed of ship in (tens) of knots.

To illustrate the use of these curves, Fig. 113 refers to an engine of 6000 I.H.P., 85 revolutions per minute, going 20 knots. The curves of diameter and pitch for a two-, three-, and four-bladed screw are plotted on an abscissæ value base. It will be noticed that the pitch curve so becomes a straight line, common to two-, three-, and four-bladed screws. For each abscissa value, the pitch is the same for two-, three-, and four-bladed screws. It will be noticed that a great variation in pitch ratio does not materially affect the efficiency.

When the ship is of exceptional form, or the problem in any way peculiar, no exact rules can be given for the proportions of screws reduced from model experiments in undisturbed water. Certainty can only be obtained by trying the model screw behind the model, as is done in the Admiralty tank.

In such a case it is better to be guided by the sea-going performance of a similar ship, because the values of the wake and propulsive coefficients may be assumed to be similar. In place of constructing constants, the following direct method may be followed:—

Let small letters refer to known ship, so that i, v, d, and r are known. Let large letters refer to new ship, so that I and V are

known, and D and R required. Then, for same pitch ratio and efficiency—

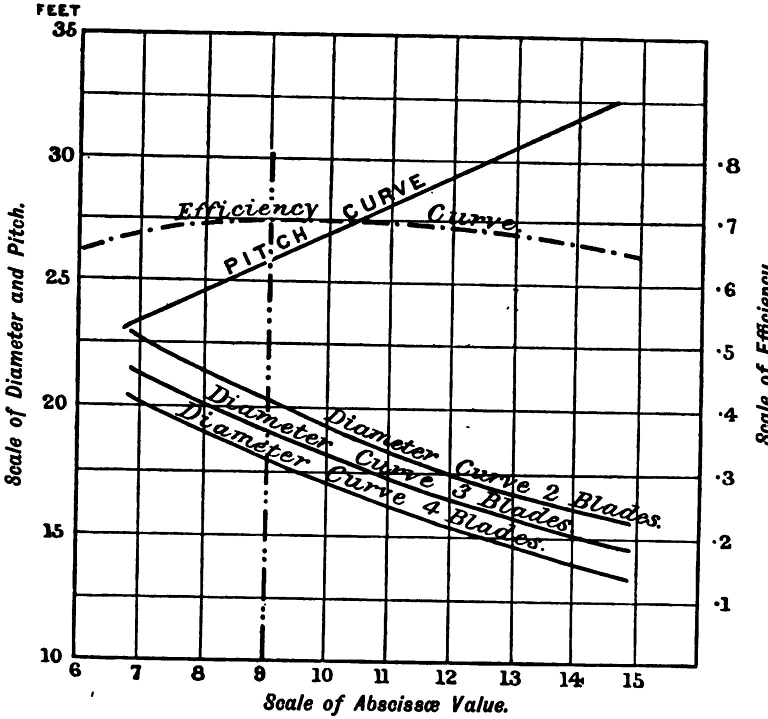

FIG. 113.

$$\frac{V^3D^2}{I} = \frac{v^3d^2}{i}$$

$$\frac{DR}{V} = \frac{dr}{v}$$

$$\text{where} \quad D = \sqrt{d^2 \times \frac{v^3}{V^3} \cdot \frac{I}{i}}$$

$$R = r\left(\frac{i}{I}\right)^{\frac{1}{2}}\left(\frac{V}{v}\right)^{2\cdot5}$$

§ 126. **Extension to Full-sized Screws.**—The results given only strictly apply to model screws. The screws experimented upon were 8-inch diameter, and the maximum speed was 206 feet

per minute. The law of comparison is assumed to apply to screws 20 or 30 feet diameter. This is a big step, but the results appear to justify the assumption. The points considered by Mr. R. E. Froude are—

(1) The hull efficiency has been assumed constant and equal to unity. For the same screw, it is approximately true for all slips, but diminution of screw diameter, with single screw, generally increases the hull efficiency by a slight amount, which is in favour of small screws. The statement applies to twin screws; for, although with twin screws, the above-mentioned influence of variation in diameter does not appear to exist, or but slightly, there is the consideration that reduction of diameter of twin screws admits of reduction of resistance of shafts, tubes, and fittings.

(2) It has been assumed that the curve of $\frac{T}{D^2V^2}$, for a given pitch ratio, say for a four-bladed screw, could be used for a three- or a two-bladed screw by altering the scale. Consequently, the maximum efficiency of two-, three-, or four-bladed screws is assumed the same. If this were true, from the last paragraph, a four-bladed screw should always be used, as giving a less diameter for given abscissa value. But experiments show that increasing the number of blades tends to decrease the maximum efficiency, so that apparently with ordinary abscissa value, there is not much to choose between two-, three-, or four-bladed screws. If limitation of diameter imposes an excessive value, the lessened abscissa value obtainable with a four-bladed screw turns the scale in its favour.

(3) The curves refer to screws of elliptical form, with the extremities of the axes at the centre and circumference of the screw respectively, the breadth of the ellipse being one-fifth the diameter of the screw. Experience shows there is no particular virtue in any particular form of blade within limits. Probably, however, the section of the blade does materially affect the results.

(4) The tables given assume a certain average value of wake factor (w) and propulsive coefficients (p), namely, $\frac{10}{9}$ and 0·5 respectively. The value of w depends mainly upon the form of

the hull and position of the screw or screws relatively to it. The standard value of w, $\frac{10}{9}$, corresponds to a wake having an absolute forward velocity of 10 per cent. of that of the ship. To use the same table for any other value of wake factor, the speed of the ship must be multiplied by $\frac{10}{9}$ and divided by the new wake factor. The multiplier for different wake factors is as follows :—

x = absolute speed of wake as a percentage of V	0	5	10	15	20	25	30
Wake factor = w	1	1·052	1·111	1·178	1·25	1·333	1·43
Multiplier of speed . . .	1·111	1·058	1·0	0·945	0·89	0·835	0·779

Ordinarily, for single-screw ships, x is 20 or 30 per cent., so that w varies from $1\frac{1}{4}$ to $1\frac{2}{5}$, and the multiplier from 0·9 to 0·8. In twin-screw ships which are carried further out, both longitudinally and athwartships, from 14 to 17 per cent., and with propellers clear of the hull, 5 to 10 per cent. In one scale for a twin-screw ship x was 30 per cent.

A variation in the propulsive coefficient is taken into account by multiplying the given I.H.P. by the new propulsive coefficient and dividing by 0·5. The variation is, however, slight.

(5) In extending the results to large screws working behind actual ships, there ought, both in the screw and in the ship, to be a correction for skin friction. Model experiments, when the results are directly extended to full-sized screws, exaggerate the effects of friction. In the actual case, there would probably be relatively less wake, but a greater efficiency of screw. The only means of obtaining data whereby the corrections can be considered, are to design a screw according to the method already given—assuming the same wake factor as is found in the model experiments for the ship considered, and to see how far actual vessels fall short of the predicted results. The analysis of such results is very difficult, but reference may be made again to Lieutenant Taylor's book on "Screw Propulsion."

(6) The above curves refer either to single or twin screws.

This would mean that single and twin screws were equally efficient. Experience shows that twin screws are not less efficient than single ones. Twin screws are better drained, they produce less augmentation of resistance, but they profit less by wake gain, they are better for steering, the screws are smaller in diameter, the revolutions are greater, and the machinery is lighter, the power is divided, and so the change of total disablement reduced. The highest propulsive coefficient yet obtained is with twin screws.

As regards triple screws,[1] the information is not sufficiently extensive to express definite opinions, but the efficiency does not appear to be materially reduced. The advantages may be taken to be those of twin screws somewhat intensified, with the addition that at cruising speed either the two side engines or the central engine may be disconnected, and so one set work at full power. The drag, or free rotation of the idle propeller, will result in a loss of power; but whether it is better to have it locked or idle can only be deduced by experiment. The results in the case of the *Greyhound* have been given. Below 10 knots, with the screw revolving, the increase in resistance was 12 to 16 per cent. in excess of that when the screw was removed; with the screw locked the resistance was about 5 to 6 per cent. less than when revolving. At 11 knots the resistance was practically the same whether the screw revolved or was locked.

(7) As regards the augmentation of resistance, in a single screw, about 20 to 40 per cent.; in twin screws, 25 per cent. in full forms to 6 per cent. in very fine forms; in torpedo boats with twin screws clear of the hull, 2 to 3 per cent.

§ 127. **Abnormal Phenomena.**—In the data already given on screw propellers it is presupposed that the screw is behind a stern having a moderately fine run, that it works well immersed, and obtains a full supply of water. If these conditions are not satisfied, the dimensions previously obtained will not apply.

In the cases of vessels with a very bluff stern the screw, instead of working in the forward wake, works in the dead water behind the stern. In such a case the supply of water to the screw is seriously impeded, and the augmentation of resistance would

[1] Admiral Melville, *Transactions of the Institution of Naval Architects*, 1899.

probably become excessive. In all cases experience is the only guide; but if the dimensions of a successful screw in one case be considered, the dimension of another might be inferred.

If the screw break the surface of the water, the efficiency of the screw may be very much reduced on account of the air being drawn in.[1] In such a case the screw might lose its hold on the water, with the result that racing occurs. The presence of air modifies the action of the screw, on account of the screw not acting on a solid, unyielding substance, but on a highly elastic mixture. If the blade of a screw is driving not simply water, but air and water, as it passes any particular portion of the mixture, the pressure will not simply drive it in front of the blade, as it would if it were a solid mass of water, but it will compress the air bubbles, which as soon as the blade has passed will expand again, driving some of the water backwards and some of it forwards. The speed for a given driving force will increase, and the air will prevent the screw clearing itself of the water it has set in motion, and will therefore tend to cause the water to whirl round the screw. But even in the case of a screw breaking the surface of the water, if the suction produced be not sufficient to cause rupture, there is no reason why the efficiency should fall off.

This racing is not confined to screws which break the surface of the water—as, for example, when a vessel pitches—but is frequently met with in screws properly immersed. In such cases the liability to race is much greater when the vessel is starting, or is towing or working against a head wind—in fact, whenever the screw is working against a resistance which reduces the speed. To understand this, the thrust of a propeller will depend on the quantity of water on which it acts in a given time, and on the sternward velocity which it imparts to this water. Now, the quantity of water on which the propeller acts is due to two causes, namely, the water which naturally flows to the screw due to the motion of the vessel, and also upon the water which the propeller itself, by its action, draws to the screw. Under ordinary circumstances the quantity due to the second cause is a small percentage

[1] Prof. Osborne Reynolds, F.R.S., *Transactions of the Institution of Naval Architects*, 1873.

of the total, but it becomes a greater and greater proportion the less the speed of the ship, and when the ship is just starting all the supply is due to the action of the screw. Thus anything, such as towing or proceeding against a head wind, that reduces the supply to the screw reduces its resistance, and consequently allays the speed of increase.

Abnormal Phenomena

§ **128. Cavitation.**—In a screw or paddle an absolute sternward velocity is impressed upon the water, and clearly the power of a propeller to supply itself with water manifestly depends on the rapidity with which fresh water will flow into the place of that which is driven astern. This rapidity depends upon whether air is present in the water or not, as the following simple case shows. Suppose a vertical plate, A, below the surface of the water, is capable of being driven forward with any velocity. If the plate were started from rest, for small velocities the water will follow the plate, but as the velocity increased the water might not keep up, and a space would be left between the plate and the following water. If the velocity was not sufficient for this, the quicker the velocity of the plate the greater would be the velocity of the following water; but, after this limit had been once exceded, the initial speed with which the water would follow would not depend on the speed of the plate, but would be the same for all speeds. This velocity is clearly that due to the head above the plate; and, taking the limiting case and assuming that the pressure behind the plate and water is reduced to zero, then the limiting velocity of the water is

$$\sqrt{2g(h + 34)}$$

h being the depth of immersion, and 34 the atmospheric head in feet of water. If the plate extends to the surface, so that air is let in behind the plate, the back of the plate is at atmospheric pressure, and the limiting velocity will then be

$$\sqrt{2gh}.$$

Thus the power of a propeller to draw water will depend on the

depth at which it acts below the water, and on whether or not the air is let in, the exclusion of air being equivalent to 34 feet additional immersion.

To apply this to a screw propeller. It has been frequently pointed out that a screw propeller acts partly by suction and partly by pressure, even in the case of what may be considered a direct-acting propeller. In addition, the rotary motion of the water behind the propeller, and the effect of suction behind the blade, causes a further suction of the water in front of the screw. Thus, the action of the screw is similar to that of a plate, only the action is more pronounced. The velocity which a propeller can produce in the water forward of the propeller is limited by the available head under which the water can flow to the propeller. The limiting case is when

$$V = \sqrt{2g(h + 34)}.$$

When this velocity is reached the screw is drawing its maximum supply of water, and a further increase in speed would only cause a cavity to be formed forward of the blades. If, when this state of affairs has been reached, air be let in behind the blades, due to a wave, the suction power would be immediately reduced by 34 feet. The resistance, therefore, would be reduced, and with same driving force, the revolutions would be very much increased. Before this velocity can be reached the reduction of pressure in front of the screw will have become so great that any air held in solution will be liberated, and, also, the water might give off vapour and so the free flow be stopped; that is, cavities would be formed.

The precise stage at which cavitation commences, with a given propeller, will depend upon the amount of acceleration which the screw tends to induce in the water in advance of the screw—cavitation taking place earlier, the greater this acceleration, and the greatest acceleration which can be permitted depending on the depth of water, the atmospheric pressure, and, to a slight extent, on temperature. But acceleration is a measure of thrusts delivered, and it is not unreasonable to expect that, with a given propeller, the stage at which cavitation commences depends on the thrust

delivered. If the thrust exceed a certain amount, the acceleration in front of the propeller will be very great, and cavitation will take place.

§ 129. Messrs. Thorneycroft and Mr. Barnaby's Experiments on Cavitation.—Cavitation was first experienced in the trials of the *Daring*.[1] With the first four screws tried (in which the blade area was about 9 square feet, and the pitch ratio varied from 1·3 to 1·46), the speed attained with a certain H.P. was 1½ knot less than that estimated. In all cases, the slip increased gradually, with speed, but at a particular speed commenced to increase very rapidly, as might have been anticipated. At this point the stern vibrated very much, although with the screws removed, the engines, at the same revolutions, failed to shake the boat. By increasing the blade area to 13 square feet, keeping the pitch and diameter practically unaltered, the slip was decreased considerably at all speeds, and cavitation was prevented. Thus, for example, increasing the surface—other things remaining unaltered—from 9 to 13 square feet, at 24 knots, caused the slip to be reduced from 30 to 17¾ per cent., and the I.H.P. from 3700 to 3050. The number of revolutions required to obtain 24 knots with the smaller surface gave 28·4 knots with the larger.

Mr. Barnaby's experiments appear to show that the limiting average pressure which can be permitted per square inch of projected blade area is, for an immersion of 1 foot, about 11¼ pounds —the projected area being taken because only the sternward component has to be considered. This was confirmed by the Hon. C. A. Parsons. Also, according to Mr. Barnaby, for each additional foot depth of immersion ⅜ pounds per square inch must be added. The pressure per square inch will be greatest at the tips.

§ 130. Hon. C. A. Parsons' Experiments on Cavitation.[2]—Mr. Parsons experienced this difficulty in an exaggerated form in the *Turbinia*, which had three shafts, with three screws on each shaft, the diameter of the screws being 2 feet, and pitch 1 foot 6 inches, and the revolutions 2200 per minute. In order to examine the

[1] Refer to the *Transactions of the Institution of Civil Engineers*, 1894; and *Transactions of the Institution of Naval Architects*, 1897.

[2] *Transactions of the Institution of Naval Architects*, 1897.

action of the screw, he experimented on a propeller 2 inches in diameter, $2\frac{1}{2}$-inch pitch, and revolving at from 1000 to 1500 revolutions, placed in a copper tank 6 inches by 6 inches cross-section, with plate-glass windows opposite the propeller. The level of the water is about 3 inches above the axis of the screw, the raised portion forming a box or vacuum space, and between it and the tank are a number of very small holes. A very high vacuum was obtained by a Fleuss compound air-pump, so that cavitation commenced at about one-tenth the revolutions under ordinary conditions. By an optical device the growth of the cavitation could be observed, and photographs taken. These photographs are taken with an exposure of 10 seconds, illumination being by a 10-ampere arc lamp ordinary 4-inch lantern condenser, throwing the beam on a concave silvered reflector mounted on the screw shaft outside the tank, the reflection being caught by another fixed concave reflector, whose centre of curvature lies on the shaft, and the beam thrown on the screw to be illuminated. The screw is thus illuminated for a small fraction of the revolution at each turn. The propeller can be observed and photographed as stationary, and the cavities in the water clearly seen and traced or photographed. It appeared that a cavity or blister first formed a little behind the leading edge (at a speed of 1000 to 1200 revolutions per minute) and near the tips of the blade; then, as the speed of revolutions increased, it enlarged in all directions until, at the speed corresponding to that in the *Turbinia's* propeller, it had grown so as to cover a section of the screw disc of 90°. When the speed was still further increased, the screw, as a whole, revolved in a cylindrical cavity, from one end of which the blades scraped off layers of solid water, delivering them one to the other. In this extreme case, nearly the whole of the energy of the screw was expended in maintaining this vacuous space. It also appeared that when the cavity had grown to be a little larger than the width of the blade, the leading edge acted like a wedge, the forward side of the edge giving negative thrust.

Thus in all screws there is a limit of speed beyond which a great loss of power results. Mr. Parsons' experience on the

Turbinia confirms Mr. Barnaby's estimate of the limiting intensity of pressure which can be permitted, namely, $11\frac{1}{4}$ pounds per square inch for each foot immersion.

By the kindness of Mr. Parsons, I am able to show two photographs (Figs. 114, 115) of—

(1) A narrow-blade propeller of ordinary proportions.

(2) A broad-blade propeller suitable for fast speeds.

§ 131. **Effect on the Design of Screw Propellers.**—In order to apply the previous results—results which state that, at a given slip and pitch ratio, the thrust varies as the square of the speed—the condition that the average thrust per square inch of projected blade area does not exceed a certain quantity must also be satisfied. With the usual rotation, and assuming a hull efficiency of unity

$$\mathrm{TV} = \rho \mathrm{V}_s$$

or

$$\mathrm{T}\,\frac{6080}{60}\,\mathrm{V} = p\,.\,\mathrm{I}\,.\,33000$$

$$\therefore \quad \mathrm{T} = \frac{60 \times 33000}{6080}\,pw\,.\,\frac{\mathrm{I}}{\mathrm{V}_s}$$

$$= 326\,pw\,\frac{\mathrm{I}}{\mathrm{V}_s}$$

p being the propulsive coefficient, w the wake factor, and I the I.H.P. per screw. If $w = \frac{10}{9}$, $p = 0{\cdot}5$, then

$$\mathrm{T} = 181\,\frac{\mathrm{I}}{\mathrm{V}_s}.$$

Let A_0, A_1, A_2, be the disc area, the combined developed area, and the combined projected blade area in square feet; and let a be the limiting intensity of pressure in pounds per square inch of projected blade area. Then

$$a\mathrm{A}_2 \times 144 = \frac{\mathrm{I}}{\mathrm{V}_s}\,326pw$$

or

$$\mathrm{A}_2 = 2{\cdot}26\,\frac{pw}{a}\,.\,\frac{\mathrm{I}}{\mathrm{V}_s}$$

which gives the minimum projected area allowed. Now A_2 is a certain fraction of A_1 depending on the pitch ratio and shape of blade; being, for a given shape, less the greater the pitch ratio.

FIG. 114.

Also A_1, for a given pattern, is a certain fraction of A_0, the fraction depending on the pattern and the number of the blades. Thence A_2 is a certain fraction of A_0, the fraction depending on the shape, pitch ratio, and number of blades. Let

$$A_2 = yA_0$$

so that

$$\text{minimum disc area} = 2{\cdot}26\,\frac{pw}{ay}\cdot\frac{I}{V_s}.$$

Thus the minimum disc area for a screw of given pattern, number of blades and pitch ratio, for a given I.H.P. and speed, is at once determined. This relation simply expresses the condition that the pressure per square inch of blade area must not exceed a pounds per square inch.

In the standard blade

$$\frac{A_1}{A_0} = 0{\cdot}2,\ 0{\cdot}3,\ 0{\cdot}4$$

according as there are two, three, or four blades. The value of $\frac{A_2}{A_1}$ may be obtained by integration for an elliptic blade. But a sufficient approximation is to take the same fraction as for a rectangular blade of the same extreme pitch ratio. In that case

$$\frac{A_2}{A_1} = \left\{\sqrt{\frac{p^2}{\pi^2}+1}-\frac{p_1}{\pi}\right\}$$

so that $y = 0{\cdot}2$, $0{\cdot}3$, or $0{\cdot}4$ times this expression according to the number of blades. For a three-bladed screw

$p_1 =$	1·0	1·5	2·0	2·5
$y =$	0·220	0·189	0·164	0·144

Using the values of O_D and O_E already given, the following table gives the limiting speed in knots for extreme pitch ratios 1·0 and 2·5, with efficiencies 0·67, 0·69, and 0·63.

FIG. 115.

Abscissa value . .		—	7	9	17
Efficiency . . .		—	0·67	0·69*f*	0·63
Speed . .	Pitch ratio	1·0	28·4	23·9	13·1
	" "	2·5	33·2	27·9	15·4

The less the abscissa value, the greater the limiting speed. Mr. Parsons recommends, for fast speeds at sea, wide, thin blades with a coarse pitch ratio.

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